1. Find two positive numbers whose product is 100 and whose sum is a minimum.

Let $x$ and $y$ be the numbers. Then $xy = 100$, so $y = \frac{100}{x}$. The sum is then

$$f(x) = x + \frac{100}{x},$$

and this is what we want to minimize. The derivative is

$$f'(x) = 1 - \frac{100}{x^2},$$

so any critical points occur when

$$1 - \frac{100}{x^2} = 0,$$

so

$$x^2 = 100,$$

or $x = \pm 10$. Since the question asks for positive numbers, we know $x > 0$, so we only need to consider the critical point $x = 10$. We want to verify that this is a minimum. Observe that the second derivative is

$$f''(x) = 1 + \frac{200}{x^3},$$

and this quantity is positive for all $x > 0$, so the function is always concave up. Therefore the critical point is a minimum. So $x = 10$ and $y = \frac{100}{10} = 10$ are the desired solutions.

2. A right circular cylinder is inscribed in a sphere of radius 1. Find the largest possible volume of such a cylinder. (Hint: Start by sketching a side view.)

Here’s a side view of the cylinder inside the sphere:
We put this circle centered at the origin, and we call the point at top right corner of the rectangle \((x,y)\). Then \(x\) is the radius of the cylinder and \(2y\) is the cylinder’s height. The volume of a cylinder is \(\pi r^2 h\), so our cylinder’s volume is \(2\pi x^2 y\). Because \(x\) and \(y\) are on the circle of radius 1, they satisfy \(x^2 + y^2 = 1\), so \(x^2 = 1 - y^2\). That allows us to write the volume as

\[
f(y) = 2\pi (1 - y^2)y = 2\pi y - 2\pi y^3.
\]

Observe also that we must have \(0 \leq y \leq 1\), so we look for the absolute maximum of \(f\) on this interval. The derivative is

\[
f'(y) = 2\pi - 6\pi y^2 = 2\pi (1 - 3y^2),
\]

and this is zero when \(y = \frac{1}{\sqrt{3}}\). (There is also a negative solution, but it’s not in the domain of our function \(f\).) We now test the critical point and the end points:

\[
f(0) = 0, \quad f(1) = 0, \quad f\left(\frac{1}{\sqrt{3}}\right) = \frac{4\pi}{3\sqrt{3}}.
\]

Therefore the largest possible volume is \(\frac{4\pi}{3\sqrt{3}}\).

\[\text{3} \quad \text{A paper cup has the shape of a cone with height 10 cm and radius 3 cm (at the top). Water is poured into the cup at a rate of 2 cm}^3/\text{sec. How fast is the water level rising when the water is 5 cm deep?}\]

Here’s a side view of the cup:

![Diagram of a paper cup with dimensions labeled](image)

The height of water in the cup is indicated as \(h\). From properties of similar triangles, we see that

\[
\frac{r}{h} = \frac{3}{10}, \quad \text{so} \quad r = \frac{3h}{10}
\]

The water in the cup occupies the shape of a cone with height \(h\) and maximum radius \(r\), so it has volume

\[
V = \frac{\pi r^2 h}{3} = \frac{\pi 3h^3}{100}.
\]

Therefore

\[
\frac{dV}{dt} = \frac{9\pi h^2 dh}{100 dt}.
\]
Plugging in $\frac{dv}{dt} = 2$ and $h = 5$ gives us

$$2 = \frac{9\pi}{4} \frac{dh}{dt},$$

so

$$\frac{dh}{dt} = \frac{8 \text{ cm}}{9\pi \text{ sec}}.$$  

Two people start from the same place at the same time, one walks north at $3 \frac{\text{miles}}{\text{hr}}$ and the other walks west at $4 \frac{\text{miles}}{\text{hr}}$. How fast is the distance between them changing after exactly 1 hour?

We note that $\frac{dx}{dt} = 4$ and $\frac{dy}{dt} = 3$. We want to find $\frac{dD}{dt}$. These quantities are related by the Pythagorean Theorem: $D^2 = x^2 + y^2$. Differentiating with respect to time yields

$$2D \frac{dD}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt}.$$  

After one hour we have $x = 4$ and $y = 3$, so $D = 5$, and plugging all this information into the previous equation gives us

$$10 \frac{dD}{dt} = 8(4) + 6(3) = 50,$$

so

$$\frac{dD}{dt} = 5 \frac{\text{miles}}{\text{hour}}.$$  

Let $f(x) = \frac{x}{x^2 + x + 1}$. Find the absolute maximum and absolute minimum values of $f$ on the interval $-2 \leq x \leq 0$.

We calculate

$$f'(x) = \frac{(x^2 + x + 1)(1) - x(2x + 1)}{(x^2 + x + 1)^2} = \frac{1 - x^2}{(x^2 + x + 1)^2}.$$  

This is zero when the numerator is zero, i.e. when $1 - x^2 = 0$, which occurs for $x = \pm 1$. Because our domain is $-2 \leq x \leq 0$, we only need to consider $x = -1$. Now we test this critical point and the endpoints:

$$f(-2) = -\frac{2}{3}, \quad f(0) = 0, \quad f(-1) = -1.$$  

Therefore the absolute maximum is 0 (which occurs at $x = 0$) and the absolute minimum is $-1$ (which occurs at $x = -1$).
6 Let \( f(x) = \frac{1}{1-x^2} \). (a) Find the vertical and horizontal asymptotes. (b) Find the intervals of increase and decrease. (c) Find the local maximum and minimum values. (d) Find the intervals of concavity. (e) Use the information from parts (a)-(d) to make a careful sketch of the graph of \( f \).

(a) 
\[
\lim_{x \to \infty} \frac{1}{1-x^2} = 0 \quad \text{and} \quad \lim_{x \to -\infty} \frac{1}{1-x^2} = 0,
\]
so \( f \) has horizontal asymptotes of \( y = 0 \) in both directions. The denominator is zero when \( x = \pm 1 \), so we check there for vertical asymptotes:

\[
\lim_{x \to 1^-} \frac{1}{1-x^2} = \infty, \quad \lim_{x \to 1^+} \frac{1}{1-x^2} = -\infty,
\]
\[
\lim_{x \to -1^-} \frac{1}{1-x^2} = -\infty, \quad \lim_{x \to -1^+} \frac{1}{1-x^2} = \infty.
\]
So \( x = 1 \) and \( x = -1 \) are both vertical asymptotes for \( f \).

(b) The derivative of \( f \) is
\[
f'(x) = \frac{2x}{(1-x^2)^2}.
\]
This is zero when \( x = 0 \), it is negative for \( x < 0 \) and positive for \( x > 0 \), so \( f \) is increasing for \( x > 0 \) and decreasing for \( x < 0 \) (except of course at the points \( x = \pm 1 \) because the function isn’t defined there).

(c) Because \( f \) changes from decreasing to increasing at \( x = 0 \), this must be a local minimum. (Note for our sketch later that \( f(0) = 1 \).)

(d) The second derivative of \( f \) is
\[
f''(x) = \frac{(1-x^2)^2(2) - 2x(2(1-x^2)(-2x))}{(1-x^2)^4}
= \frac{(1-x^2)(2) - 2x(2(-2x))}{(1-x^2)^3}
= \frac{2+6x^2}{(1-x^2)^3}.
\]
Notice that the numerator is always positive. Meanwhile, the denominator is positive when \(-1 < x < 1 \) and negative otherwise. Therefore \( f \) is concave up when \(-1 < x < 1 \) and negative otherwise.