Show all your work. You must show all the steps to get full credit for a solution. When you need to calculate a derivative, use the techniques from Chapter 3 – do not attempt to calculate the derivatives directly using limits.

1. (4 points) Find the linearization (also called a linear approximation or tangent-line approximation) for the function \( f(x) = \sqrt{x} \) at the point \( x = 9 \) and use it to estimate the value of \( \sqrt{10} \).

\[
\begin{align*}
  f'(x) &= \frac{1}{2\sqrt{x}} \\
  f'(9) &= \frac{1}{2\sqrt{9}} = \frac{1}{6} \\
  f'(9) &= 3
\end{align*}
\]

So the tangent line is

\[
\begin{align*}
  y - 3 &= \frac{1}{6}(x - 9) \\
  y &= \frac{1}{6}x - \frac{9}{6} + 3 \\
  &= \frac{1}{6}x + \frac{9}{6}.
\end{align*}
\]

So \( L_9(x) = \frac{1}{6}x + \frac{9}{6} \).

Therefore, \( \sqrt{10} = f(10) \)

\[
\begin{align*}
  \approx L_9(10) \\
  &= \frac{10}{6} + \frac{9}{6} \\
  &= \frac{19}{6} \\
  L_9(x) &= \frac{1}{6}x + \frac{9}{6} \\
  \sqrt{10} &\approx \frac{19}{6}
\end{align*}
\]
The figure shows the graph of the ellipse $x^2 + 2y^2 = 3$.

(a) **(4 points)** Find the slope of a tangent line at the point $(1, 1)$.

\[
\frac{d}{dx} \left[ x^2 + 2y^2 \right] = \frac{d}{dx} [3]
\]

\[2x + 4y \frac{dy}{dx} = 0\]

\[4y \frac{dy}{dx} = -2x\]

\[\frac{dy}{dx} = \frac{-x}{2y}\]

At $x = 1, y = 1$, we have $\frac{dy}{dx} = \frac{-1}{2}$.

Slope: $\frac{-1}{2}$

(b) **(3 points)** Find all the coordinates of points on the ellipse where the slope of the tangent line is exactly 1.

We want to have $\frac{dy}{dx} = 1$, so $\frac{-x}{2y} = 1$, thus $x = -2y$.

Plug this into the equation for the ellipse to get

\[(-2y)^2 + 2y^2 = 3 \quad \Rightarrow \quad 4y^2 + 2y^2 = 3\]

\[\Rightarrow 6y^2 = 3 \quad \Rightarrow \quad y^2 = \frac{1}{2} \quad \Rightarrow \quad y = \pm \frac{1}{\sqrt{2}}.\]

Now use $x = -2y$ to get the x-coordinates:

- If $y = \frac{1}{\sqrt{2}}$, $x = \frac{-2}{\sqrt{2}} = -\sqrt{2}$
- If $y = -\frac{1}{\sqrt{2}}$, $x = \frac{2}{\sqrt{2}} = \sqrt{2}$

Coordinates: $\left( \sqrt{2}, \frac{1}{\sqrt{2}} \right)$ and $\left( -\sqrt{2}, -\frac{1}{\sqrt{2}} \right)$. 
3 (5 points) Find the exact absolute maximum and minimum values of the function \( f(x) = x^3 - 12x \) on the interval \( 0 \leq x \leq 3 \).

\[
\frac{df}{dx} = 3x^2 - 12 = 3(x^2 - 4) = 3(x + 2)(x - 2).
\]
This is zero when \( 3(x + 2)(x - 2) = 0 \), so \( x = 2 \) and \( x = -2 \) are the critical points. Only \( x = 2 \) is in the interval \( 0 \leq x \leq 3 \), so we check this and the endpoints:

\[
\begin{align*}
\text{abs. max} & : f(0) = 0 \\
\text{abs. min} & : f(2) = -16 \\
\text{abs.} & : f(3) = -9
\end{align*}
\]

maximum value: \( 0 \)

minimum value: \( -16 \)

4 (4 points) Prove that \( \frac{d}{dx} \tan x = \sec^2 x \) using the derivatives of sine and cosine.

\[
\begin{align*}
\frac{d}{dx} \tan x & = \frac{d}{dx} \left[ \frac{\sin x}{\cos x} \right] \\
& = \frac{\cos x \frac{d}{dx} [\sin x] - \sin x \frac{d}{dx} [\cos x]}{\cos^2 x} \\
& = \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x} \\
& = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\
& = \frac{1}{\cos^2 x} \\
& = \sec^2 x
\end{align*}
\]
5 (5 points) The figure at right shows the curve \( y = x^2 \) and a tangent line with positive slope that passes through the point with coordinates \((0, -5)\). Find the exact equation of the tangent line.

We need to determine the point \((x, x^2)\) where the tangent line touches the parabola.

Since the line also goes through \((0, -5)\), the slope is \( \frac{x^2 - (-5)}{x - 0} = \frac{x^2 + 5}{x} \).

Since the line is tangent, the slope is \( y'(x) = 2x \).

Therefore

\[
\frac{x^2 + 5}{x} = 2x \quad \Rightarrow \quad x^2 + 5 = 2x^2 \\
\Rightarrow \quad 5 = x^2 \\
\Rightarrow \quad x = \pm \sqrt{5}.
\]

Since we want the line with positive slope, it must be \( x = \sqrt{5} \) when the line touches the parabola.

At that point, \( y = (\sqrt{5})^2 = 5 \), and the slope is \( y' = 2\sqrt{5} \). So the tangent line is

\[
y - 5 = 2\sqrt{5} (x - \sqrt{5})
\]

or

\[
y = 2\sqrt{5}x - 5
\]

Equation: \( y = 2\sqrt{5}x - 5 \)
(4 points) Find all of the critical points of the function \( y = \sin x + \sqrt{3} \cos x \). Give exact answers, not decimal approximations.

\[
\frac{dy}{dx} = \cos x - \sqrt{3} \sin x = 0 \quad \text{when}
\]

\[
\cos x = \sqrt{3} \sin x
\]

\[
\Rightarrow \frac{1}{\sqrt{3}} = \frac{\sin x}{\cos x} = \tan x.
\]

From a reference triangle, we see that \( x = \frac{\pi}{6} \) is a solution.

From a graph of \( y = \tan x \) and knowing that the period is \( \pi \), we see that the general solution is \( x = \frac{\pi}{6} + k\pi \), where \( k \) is any integer.

\[
x = \frac{\pi}{6} + k\pi
\]

(3 points) Suppose \( y = x^x \). Find the slope of the tangent line at the point where \( x = e \).

\[
\ln y = x \ln x
\]

\[
\frac{1}{y} \frac{dy}{dx} = x \cdot \frac{1}{x} + \ln x = 1 + \ln x
\]

\[
\frac{dy}{dx} = y(1 + \ln x)
\]

\[
\frac{dy}{dx} = x^x(1 + \ln x)
\]

At the point where \( x = e \), we get

\[
\frac{dy}{dx} = e^e (1 + \ln e) = 2e^e
\]

slope: \( 2e^e \)