Worksheet #7 - Error Analysis

In this worksheet, you will identify the errors in proposed solutions to calculus problems.

**INSTRUCTIONS:** In each of the problems below, identify the errors or the nature of the misunderstandings in the proposed solutions.

1. Find the derivative of \( y = x^x \).
   **Solution:** We use logarithmic differentiation:
   
   \[
   \ln y = \ln x^x \\
   \implies \ln y = x \ln x \\
   \implies \frac{1}{y} \frac{dy}{dx} = \frac{d}{dx}[x \ln x] = 1 \left( \frac{1}{x} \right) \\
   \implies \frac{dy}{dx} = \frac{y}{x} = \frac{x^x}{x} = x^{x-1}.
   \]

2. Find the equation of the tangent line to \( y = \sin(3x) \) at the point where \( x = \frac{\pi}{3} \).
   **Solution:** First we find the slope using the derivative.
   
   \[ y' = \cos(3x). \]
   
   When \( x = \frac{\pi}{3} \) we get \( y = \sin(\pi) = 0 \). Therefore the equation of the tangent line is given by
   
   \[ y - 0 = \cos(3x)(x - \frac{\pi}{3}) \text{ or } y = \cos(3x)(x - \frac{\pi}{3}). \]

3. Find the equation of a tangent line to the graph of \( y = x^3 \) that passes through the point \((1, 2)\).
   **Solution:** The derivative is \( y' = 3x^2 \). So when \( x = 1 \) we have \( y' = 3 \). Therefore the tangent line is given by
   
   \[ y - 2 = 3(x - 1) \text{ or } y = 3x + 1. \]
4] Find the exact value of the absolute maximum of \( y = x^3 - x \) on the interval \([-2, 2]\).

**Solution:** The derivative is \( y' = 3x^2 - 1 \). Set this equal to zero to locate the critical points:

\[
3x^2 - 1 = 0 \implies 3x^2 = 1 \implies x^2 = \frac{1}{3} \implies x = \pm \frac{1}{\sqrt{3}}.
\]

So the critical points are \( x = \frac{1}{\sqrt{3}} \) and \( x = -\frac{1}{\sqrt{3}} \). Now we evaluate the function at these points:

\[
f\left(\frac{1}{\sqrt{3}}\right) = -0.3849
\]

and

\[
f\left(-\frac{1}{\sqrt{3}}\right) = 0.3849
\]

Therefore the absolute maximum is 0.3849.

5] Calculate slope of tangent line to \( y = \sqrt{x^2 + x^4} \) at the point where \( x = 1 \).

**Solution:** First we take a derivative:

\[
y' = \frac{d}{dx}\left[\sqrt{x^2 + x^4}\right] = \frac{d}{dx}\left[x + x^2\right] = 1 + 2x.
\]

Therefore \( y'(1) = 3 \).

6] Find all the critical points of \( \sin(3x) \).

**Solution:** Let \( y = \sin(3x) \). Then

\[
y' = \cos(3x).
\]

The critical points occur when the derivative is zero, so:

\[
\cos(3x) = 0 \implies 3x = \cos^{-1}(0) = 1.57 \implies x = 0.5236.
\]

7] Find the inflection points of \( y = \frac{x^2}{x+1} \).

**Solution:**

\[
y' = \frac{(x+1)(2x) - (x^2)(1)}{(x+1)^2} = \frac{2x^2 - x^2}{x^2 + 2x + 1} = \frac{x^2}{x^2 + 2x + 1} = \frac{1}{2x+1}.
\]

But \( \frac{1}{2x+1} \) can never be zero because the numerator is never zero, so there are no inflection points for this function.
Find the $x$ and $y$ coordinates points the points on the ellipse $x^2 + 2y^2 = 4$ where the slope of the tangent line is 2.

**Solution:** We can use implicit differentiation:

$$x^2 + 2y^2 = 4$$

$$\implies \frac{d}{dx}[x^2 + 2y^2] = 4$$

$$\implies 2x + 4y \frac{dy}{dx} = 4$$

$$\implies 4y \frac{dy}{dx} = 4 - 2x$$

$$\implies \frac{dy}{dx} = \frac{4 - 2x}{4} = 1 - \frac{x}{2}.$$

So we need to solve

$$1 - \frac{x}{2} = 2.$$

We obtain

$$-\frac{x}{2} = -1 \implies x = 2.$$

Find the values of $x$ where the graph of $y = \frac{x}{2x+1}$ has a tangent line that is perpendicular to the line $y = 4x$.

**Solution:** We need to know when the slope of the tangent line is $\frac{1}{4}$. So we find the derivative, set it equal to this and solve the equation.

$$y' = \frac{x(2) - (2x + 1)(1)}{(2x + 1)^2}$$

$$= \frac{2x - 2x + 1}{4x^2 + 1}$$

$$= \frac{1}{4x^2 + 1}.$$

So we have to solve

$$\frac{2x - 2x + 1}{4x^2 + 1} = \frac{1}{4}.$$

Cross multiply to get

$$4 = 4x^2 + 1 \implies 3 = 4x^2 \implies x^2 = \frac{3}{4} \implies x = \pm \frac{\sqrt{3}}{2}.$$