Solutions to Sample Questions for Exam #1

1 Write down a Riemann Sum with \( n \) rectangles for the area above the \( x \)-axis and under the curve \( y = \sin(x) \) for \( 0 \leq x \leq \frac{\pi}{2} \). Use right endpoints and \( \Sigma \) notation for your answer.

The interval \( 0 \leq x \leq \frac{\pi}{2} \) has length \( \frac{\pi}{2} \), so each subinterval will have length

\[ \Delta x = \frac{\pi}{2n} \]

The right endpoint of the \( i^{th} \) subinterval is

\[ x_i = i\Delta x = \frac{\pi i}{2n} \]

Thus the Riemann sum is

\[ \sum_{i=1}^{n} f(x_i) \Delta x = \sum_{i=1}^{n} \sin \left( \frac{\pi i}{2n} \right) \frac{\pi}{2n} \]

2 Estimate the area under the graph of \( y = \ln x \) and above the \( x \)-axis for \( 1 \leq x \leq 2 \) using 4 subintervals and (a) right endpoints as sample points, and (b) left endpoints as sample points. Include 3 decimal places in your answers.

(a) The width of the entire interval is 1, so the width of each subinterval is \( \Delta x = \frac{1}{4} \). The right endpoints of the subintervals are \( x_1 = \frac{5}{4} \), \( x_2 = \frac{3}{2} \), \( x_3 = \frac{7}{4} \) and \( x_4 = 2 \). Therefore, the areas of the approximating rectangles are

\[ A_1 = \frac{1}{4} \ln \frac{5}{4}, \quad A_2 = \frac{1}{4} \ln \frac{3}{2}, \quad A_3 = \frac{1}{4} \ln \frac{7}{4}, \quad \text{and} \quad A_4 = \frac{1}{4} \ln 2 \]

Therefore the total area is approximately

\[ A_1 + A_2 + A_3 + A_4 = \frac{1}{4} \ln \frac{5}{4} + \frac{1}{4} \ln \frac{3}{2} + \frac{1}{4} \ln \frac{7}{4} + \frac{1}{4} \ln 2 \approx 0.470 \]

(b) The width of the entire interval is 1, so the width of each subinterval is \( \Delta x = \frac{1}{4} \). The left endpoints of the subintervals are \( x_1 = 1 \), \( x_2 = \frac{5}{4} \), \( x_3 = \frac{3}{2} \) and \( x_4 = \frac{7}{4} \). Therefore, the areas of the approximating rectangles are

\[ A_1 = \frac{1}{4} \ln 1, \quad A_2 = \frac{1}{4} \ln \frac{5}{4}, \quad A_3 = \frac{1}{4} \ln \frac{3}{2}, \quad \text{and} \quad A_4 = \frac{1}{4} \ln \frac{7}{4} \]

Therefore the total area is approximately

\[ A_1 + A_2 + A_3 + A_4 = 0 + \frac{1}{4} \ln \frac{5}{4} + \frac{1}{4} \ln \frac{3}{2} + \frac{1}{4} \ln \frac{7}{4} \approx 0.297 \]
3 Calculate $\int_4^9 \frac{3x^2 - 2}{\sqrt{x}} \, dx$. Show all your work.

$$\int_4^9 \frac{3x^2 - 2}{\sqrt{x}} \, dx = \int_4^9 \frac{3x - 2}{\sqrt{x}} \, dx$$
$$= \int_4^9 3x^{\frac{1}{2}} - 2x^{-\frac{1}{2}} \, dx$$
$$= \left[ 2x^{\frac{3}{2}} - 4x^{\frac{1}{2}} \right]_4^9$$
$$= (2 \cdot 9^{\frac{3}{2}} - 4 \cdot 9^{\frac{1}{2}}) - (2 \cdot 4^{\frac{3}{2}} - 4 \cdot 4^{\frac{1}{2}})$$
$$= (54 - 12) - (16 - 8)$$
$$= 34.$$ 

4 Calculate $\int x^3 \cos (x^4) \, dx$. Show all your work.

$$\int x^3 \cos (x^4) \, dx = \int \cos u \, \frac{du}{4} \quad \left( u = x^4, \, \frac{du}{4} = x^3 \, dx \right)$$
$$= \frac{1}{4} \int \cos u \, du$$
$$= \frac{1}{4} \sin u + C$$
$$= \frac{1}{4} \sin x^4 + C.$$ 

5 Calculate $\int \cot \theta \, d\theta$ using substitution. Show all your work.

$$\int \cot \theta \, d\theta = \int \cot \theta \, \frac{d\theta}{\sin \theta}$$
$$= \int \frac{1}{u} \, du \quad \left( u = \sin \theta, \, du = \cos \theta \, d\theta \right)$$
$$= \ln |u| + C$$
$$= \ln |\sin \theta| + C.$$
[6] Calculate \( \int_1^4 \sqrt{t} \ln t \, dt \). Show all your work.

\[
\int \sqrt{t} \ln t \, dt = \int t^{\frac{1}{2}} \ln t \, dt
\]

\[
= \ln t \left( \frac{2}{3} t^{\frac{3}{2}} \right) - \int \left( \frac{2}{3} t^{\frac{3}{2}} \right) \frac{1}{t} \, dt
\]

\[
= \frac{2}{3} t^{\frac{3}{2}} \ln t - \frac{2}{3} \int t^{\frac{1}{2}} \, dt
\]

\[
= \frac{2}{3} t^{\frac{3}{2}} \ln t - \frac{4}{9} t^{\frac{3}{2}}
\]

Therefore

\[
\int_1^4 \sqrt{t} \ln t \, dt = \left. \frac{2}{3} t^{\frac{3}{2}} \ln t - \frac{4}{9} t^{\frac{3}{2}} \right|_1^4
\]

\[
= \left( \frac{2}{3} \cdot 4^{\frac{3}{2}} \ln 4 - \frac{4}{9} \cdot 4^{\frac{3}{2}} \right) - \left( \frac{2}{3} \cdot 1^{\frac{3}{2}} \ln 1 - \frac{4}{9} \cdot 1^{\frac{3}{2}} \right)
\]

\[
= \frac{16 \ln 4}{3} - \frac{28}{9}
\]

[7] Calculate \( \int x^2 e^x \, dx \). Show all your work.

\[
\int x^2 e^x \, dx = x^2 e^x - \int 2x e^x \, dx
\]

\[
= x^2 e^x - \left( 2x e^x - \int 2 e^x \, dx \right)
\]

\[
= x^2 e^x - (2x e^x - 2e^x) + C
\]

\[
= x^2 e^x - 2x e^x + 2e^x + C
\]
8 Use partial fractions to evaluate \( \int \frac{1}{x^2+4x+3} \, dx \). Show all your work.

The denominator \( x^2 + 4x + 3 \) can be factored as \((x + 3)(x + 1)\), so we want to decompose the fraction as follows:

\[
\frac{1}{(x + 3)(x + 1)} = \frac{A}{x + 3} + \frac{B}{x + 1}
\]

Multiply both sides by the denominator of the left to get

\[
1 = A(x + 1) + B(x + 3).
\]

If we plug in \( x = -1 \) we get the equation

\[
1 = 2B,
\]

so \( B = \frac{1}{2} \). If we plug in \( x = -3 \) we get the equation

\[
1 = -2A,
\]

so \( A = -\frac{1}{2} \). That is to say,

\[
\frac{1}{(x + 3)(x + 1)} = \frac{-\frac{1}{2}}{x + 3} + \frac{\frac{1}{2}}{x + 1}.
\]

Consequently,

\[
\int \frac{1}{x^2 + 4x + 3} \, dx = \int \frac{1}{(x + 3)(x + 1)} \, dx
\]

\[
= \int \frac{-\frac{1}{2}}{x + 3} + \frac{\frac{1}{2}}{x + 1} \, dx
\]

\[
= -\frac{1}{2} \int \frac{1}{x + 3} \, dx + \frac{1}{2} \int \frac{1}{x + 1} \, dx
\]

\[
= -\frac{1}{2} \int ds + \frac{1}{2} \int dt \quad (s = x + 3, \; ds = dx \text{ and } t = x + 1, \; dt = dx)
\]

\[
= -\frac{1}{2} \ln |s| + \frac{1}{2} \ln |t| + C
\]

\[
= -\frac{1}{2} \ln |x + 3| + \frac{1}{2} \ln |x + 1| + C.
\]

9 First use substitution, then integration-by-parts, to calculate the integral \( \int \cos \sqrt{x} \, dx \).

\[
\int \cos \sqrt{x} \, dx = \int (\cos s)2s \, ds \quad (s = \sqrt{x}, \; ds = \frac{1}{2\sqrt{x}} \, dx, \text{ or } 2s \, ds = dx)
\]

\[
= 2s \sin s - \int 2 \sin s \, ds \quad (u = 2s \quad dv = \cos s \, ds)
\]

\[
= 2s \sin s + 2 \cos s + C
\]

\[
= 2\sqrt{x} \sin \sqrt{x} + 2 \cos \sqrt{x} + C.
\]
10. Calculate the area under the curve $y = x^2$ and above the $x$-axis, for $0 \leq x \leq 3$, by calculating a limit of Riemann Sums. You may use the following formula:

$$\sum_{j=1}^{N} j^2 = \frac{N(N + 1)(2N + 1)}{6}.$$ 

With $N$ rectangles, the length of each subinterval is $\Delta x = \frac{3}{N}$. Using right endpoints, the sample points will be $x_j = \frac{3j}{N}$. Therefore the area is equal to

$$\lim_{N \to \infty} \sum_{j=1}^{n} f(x_j) \Delta x = \lim_{N \to \infty} \sum_{j=1}^{n} \left( \frac{3j}{N} \right)^2 \frac{3}{N} = \lim_{N \to \infty} \sum_{j=1}^{n} \frac{27j^2}{N^3} = \lim_{N \to \infty} \frac{27}{N^3} \sum_{j=1}^{n} j^2 = \lim_{N \to \infty} \frac{27N(N+1)(2N+1)}{N^3} \cdot \frac{6}{6} = \lim_{N \to \infty} \frac{18N^2 + 27N + 9}{2N^2} = 9.$$

11. A function $f(x)$ is defined on the interval $(0, \infty)$ by the formula

$$f(x) = \int_{0}^{x^2} \frac{\sin(t)}{t} dt.$$ 

Find all the critical points of this function.

According to the Fundamental Theorem of Calculus,

$$\frac{d}{dx} \int_{0}^{x^2} \frac{\sin(t)}{t} dt = \frac{\sin(x^2)}{x^2} \frac{d}{dx} (x^2) = \frac{\sin(x^2)}{x^2} (2x) = \frac{2\sin(x^2)}{x}.$$ 

So the critical points occur where this is zero:

$$\frac{2\sin(x^2)}{x} = 0 \implies \sin(x^2) = 0 \implies x^2 = \pi k \implies x = \sqrt{\pi k},$$

where $k$ is any positive integer.