Worksheet #8 - Separation of Variables

In this worksheet, you will use the method of separation of variables to solve differential equations.

1. Consider the differential equation

\[ \frac{dy}{dx} = y^2 \sin x. \]

(a) Find the general solution of this differential equation.

(b) Find the particular solution that also satisfies the condition \( y(0) = 2 \).
Newton’s Law of Cooling states that the rate of change in temperature of an object is proportional to the temperature difference between the object and its surroundings.

(a) Use Newton’s Law of Cooling to write down a differential equation for the temperature $T$ at time $t$ for an object surrounded by a temperature $T_s$. (Your differential equation will have $T$, $\frac{dT}{dt}$, $T_s$ and another constant, call it $k$, that represents the constant of proportionality.)

(b) Solve the differential equation you found in part (a), treating $k$ and $T_s$ as constants.
(c) Suppose a turkey is removed from the oven and its temperature is measured to be 175 degrees (this is the initial temperature, \(u(0) = 175\)). The room temperature is 65 degrees. After 15 minutes, the temperature is measured to be 165 degrees. Find a formula for the temperature of the turkey at time \(t\). (Hint: You can just plug \(T_s = 65\) into the formula you found in part (b), but you’ll have to set up a system of equations to find the other unknowns.)

(d) You want to serve the turkey when it reaches 150 degrees. How much longer will that take?
A population of size $P(t)$ at time $t$ can’t realistically grow forever—it will be limited by the carrying-capacity of the environment. Call this number $K$—it’s the maximum population size the environment can sustain. Call the natural growth rate of the population $k$—this is the growth rate that the population would exhibit if it had unlimited resources. The differential equation that models this population’s growth is

$$\frac{dP}{dt} = kP \left(1 - \frac{P}{K}\right).$$

This is called the logistic equation. This idea is that the quantity in parentheses will be small when $P$ is close to the carrying capacity $K$ because the environment’s resources be in scarce supply. Thus the term in parentheses makes the growth rate $\frac{dP}{dt}$ small when $P$ is close to $K$.

(a) Find the general solution to the logistic equation, treating $k$ and $K$ as constants. (Hint: Write the term inside parentheses as a single quotient. You will also need to use partial fractions.)

(b) What is $\lim_{t \to \infty} P(t)$?