1. (a) \[ \sum_{n=1}^{\infty} \left| \frac{n(-1)^n}{n^2 + 1} \right| = \sum_{n=1}^{\infty} \frac{n}{n^2 + 1} \]

\[ \lim_{n \to \infty} \left( \frac{n}{n^2 + 1} \right) = \lim_{n \to \infty} \frac{n^2}{n^2 + 1} = 1, \]

and \( \sum_{n=1}^{\infty} \frac{1}{n} \) diverges.

So, by the Limit Comparison Test, \( \sum_{n=1}^{\infty} \frac{n}{n^2 + 1} \) diverges.

So \( \sum_{n=1}^{\infty} \frac{n(-1)^n}{n^2 + 1} \) does not converge absolutely.

On the other hand,

\[ \lim_{n \to \infty} \frac{n}{n^2 + 1} = 0, \quad \frac{n}{n^2 + 1} \text{ is nonnegative, and } \frac{n}{n^2 + 1} \]

is decreasing since

\[ \frac{d}{dx} \left[ \frac{x}{x^2 + 1} \right] = \frac{(x^2 + 1)(1) - x(2x)}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2} \leq 0 \text{ for } x > 1. \]

So, according to the Alternating Series Test, \( \sum_{n=1}^{\infty} \frac{n(-1)^n}{n^2 + 1} \) converges conditionally.
(b) \[ \sum_{n=1}^{\infty} \left| \frac{(-1)^n}{\sqrt{n^3+1}} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}} \]

We'll use a Limit Comparison Test with \[ \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}. \]

\[
\lim_{n \to \infty} \frac{\frac{1}{\sqrt{n^3+1}}}{\left(\frac{1}{n^{3/2}}\right)} = \lim_{n \to \infty} \frac{n^{3/2}}{\sqrt{n^3+1}}
\]

\[
= \lim_{n \to \infty} \sqrt{\frac{n^3}{n^3 + 1}}
\]

\[= \sqrt{1} = 1.\]

Since \[ \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \] converges, so does \[ \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}} \].

Therefore, \[ \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^3+1}} \] is absolutely convergent.

\[ \square \]

We will start by applying the Ratio Test:

\[ a_n = \frac{(-1)^n X^{2n}}{n+2} \]

\[ a_{n+1} = \frac{(-1)^{n+1} X^{2(n+1)}}{(n+1)+2} = \frac{(-1)^n X^{2n+2}}{n+3} \]
\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|
\]
\[
= \lim_{n \to \infty} \left| \frac{(1)^{n+1} x^{2n+2}}{n+3} \cdot \frac{n+2}{(1)^n x^{2n}} \right|
\]
\[
= \lim_{n \to \infty} \left| \frac{(-1) x^2 (n+2)}{n+3} \right|
\]
\[
= \lim_{n \to \infty} |x^2| \cdot \frac{n+2}{n+3}
\]
\[
= |x|^2.
\]

Therefore the series converges absolutely for
\[
|X|^2 < 1
\]
i.e. for \(-1 < x < 1\).

Next, we need to check the endpoint of this interval.

**Endpoint:** \( x = 1 \)

\[
\sum_{n=0}^{\infty} \frac{(-1)^n}{n+2}
\]

is a convergent alternating series, but it does not converge absolutely because \( \sum_{n=0}^{\infty} \frac{1}{n+2} \) diverges.
End point: $x = -1$

\[ \sum_{n=0}^{\infty} \frac{(-1)^n}{n+2}, \text{ just like above, converges conditionally.} \]

Therefore,

\[ \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n+2} \]

converges absolutely for $-1 < x < 1$

and converges conditionally for $x = -1, x = 1$.

\[ \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \]

\[ \Rightarrow \quad \frac{d}{dx} \left[ \frac{1}{1-x} \right] = \frac{d}{dx} \sum_{n=0}^{\infty} x^n \]

\[ \Rightarrow \quad \frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} nx^{n-1} \]

\[ \Rightarrow \quad \frac{x^3}{(1-x)^2} = x^3 \sum_{n=0}^{\infty} nx^{n-1} \]

\[ \Rightarrow \quad \frac{x^3}{(1-x)^2} = \sum_{n=0}^{\infty} nx^{n+2} \]
If we plug in $x = \frac{1}{3}$, the series becomes

$$\sum_{n=0}^{\infty} n \left(\frac{1}{3}\right)^{n+2} = \sum_{n=0}^{\infty} \frac{n}{3^{n+2}}.$$  

Since the first term of this series is 0, this is really the same as

$$\sum_{n=1}^{\infty} \frac{n}{3^{n+2}}.$$  

Therefore, we have

$$\sqrt{\sum_{n=1}^{\infty} \frac{n}{3^{n+2}}} = \frac{\left(\frac{1}{3}\right)^3}{(1 - \frac{1}{3})^2} = \frac{\left(\frac{1}{3}\right)^3}{\left(\frac{2}{3}\right)^2} = \frac{1}{27} \cdot \frac{9}{4} = \frac{1}{12}.$$