Worksheet #2 - Sequences

In this worksheet, you will analyze sequences of real numbers.

1. Consider the sequence given by the formula $a_n = \frac{2n+1}{3n-1}$.
   
   (a) Write out the first 5 terms of the sequence in the form $\{a_1, a_2, a_3, a_4, a_5, \ldots\}$.

   (b) Create a dot plot of the sequence by carefully plotting each of the points $(1, a_1), (2, a_2)$, etc. on an xy-plane.

   (c) The dot plot should suggest that the sequence $\{a_n\}$ is decreasing. Prove that this is actually the case by proving that the function $f(x) = \frac{2x+1}{3x-1}$ is decreasing for all $x \geq 1$. (Hint: Use differentiation.)

   (d) Calculate the limit of the sequence $\{a_n\}$. 
Consider the sequence given by the formula $b_n = \frac{n+1}{n^2+2}$.

(a) Write out the first 5 terms of the sequence in the form \( \{b_1, b_2, b_3, b_4, b_5, \ldots \} \).

(b) Create a dot plot of the sequence by carefully plotting each of the points \((1, b_1), (2, b_2),\) etc. on an \(xy\)-plane.

(c) The dot plot should suggest that the sequence \( \{b_n\} \) is decreasing. Prove that this is actually the case by proving that the function \( g(x) = \frac{x+1}{x^2+2} \) is decreasing for all \( x \geq 1 \). (This will require some explanation to go along with the calculations.)

(d) Calculate the limit of the sequence \( \{b_n\} \).
Consider the sequence given by \( a_n = \frac{1+(-1)^n}{n+2} \).

(a) Write out the first 5 terms of the sequence in the form \( \{a_1, a_2, a_3, a_4, a_5, \ldots\} \).

(b) Create a dot plot of the sequence by carefully plotting each of the points \((1, a_1), (2, a_2), \) etc. on an \(xy\)-plane.

The dot plot should show that the sequence \( \{a_n\} \) is neither increasing nor decreasing. Also, because the base \((-1)\) in the exponential term is negative, our normal differentiation formulas don’t work for this sequence. (In particular, L’Hospital’s Rule will not work well with this sequence.) So, in order to determine the limit, we will use a high-tech approach: the Squeeze Theorem.

(c) Prove that \( 0 \leq a_n \leq \frac{2}{n+2} \) for all positive integers \( n \). (Hint: There are two cases: when \( n\) is even and when it is odd.)

(d) Use the Squeeze Theorem to calculate the limit of the sequence \( \{a_n\} \).
Let \( s_n = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots + \frac{1}{3^n} \).

(a) Write out the first 5 terms of the sequence in the form \( \{s_1, s_2, s_3, s_4, s_5, \ldots \} \).

(b) Prove that \( 3s_n - s^n = 1 - \frac{1}{3^n} \) for every positive integer \( n \). (Hint: Don’t simplify this to \( 2s_n \). Instead, write out \( 3s_n \) and write out \( s^n \), then do some simplifying.)

(c) The step above proved that \( 2s_n = 1 - \frac{1}{3^n} \). Use this fact to calculate \( \lim_{n \to \infty} s_n \).