Find the points on the surface \( xy^2 z^3 = 2 \) that are closest to the origin. We use the method of Lagrange Multipliers, with \( f(x, y, z) = x^2 + y^2 + z^2 \) for the square of the distance from the origin, and the constraint is given by \( g = 2 \) for \( g(x, y, z) = xy^2 z^3 \). We get
\[
\nabla f = (2x, 2y, 2z) \quad \text{and} \quad \nabla g = (y^2 z^3, 2xyz^3, 3xy^2 z^2).
\]

Setting \( \nabla f = \lambda \nabla g \) gives us the system of equations
\[
\begin{align*}
2x &= \lambda y^2 z^3 \\
2y &= 2\lambda xyz^3 \\
2z &= 3\lambda xy^2 z^2 \\
xy^2 z^3 &= 2.
\end{align*}
\]

Note that the last equation implies that \( x, y, z \neq 0 \). Dividing both sides of the second equation by \( 2y \) gives \( 1 = \lambda xz^3 \), so \( \lambda = \frac{1}{xz^3} \). Plug that into the first equation and clear fractions to obtain
\[
2x^2 = y^3.
\]

Case 1: \( z = \sqrt{3}x \). This and \( y^2 = 2x^2 \) allow us to write the constraint equation entirely in terms of \( x \):
\[
6\sqrt{3}x^6 = 2 \implies x^6 = 3^{3/2} \implies x = \pm 3^{-1/4}.
\]

Case 2: \( z = -\sqrt{3}x \). That implies \( x \) and \( z \) have opposite signs. Therefore \( xy^2 z^3 \) is negative, since \( y^2 \) is positive. This is not a possibility since our constraint equation requires \( xy^2 z^3 = 2 \). Therefore Case 2 does not occur.

We also have \( y^2 = 2x^2 = 2(3^{-1/4})^2 = \frac{2}{\sqrt{3}} \), so \( y = \pm \frac{\sqrt{3}}{3^{1/4}} \).

Therefore, the solutions of the system of equations above are
\[
(3^{-1/4}, 2(3^{-1/4}), 3^{1/4}), \quad (-3^{-1/4}, -2(3^{-1/4}), 3^{1/4}),
\]
\[
(-3^{-1/4}, 2(3^{-1/4}), -3^{1/4}) \quad \text{and} \quad (-3^{-1/4}, -2(3^{-1/4}), -3^{1/4}).
\]

Every one of these points has the same distance from the origin:
\[
\sqrt{(\pm 3^{-1/4})^2 + (\pm 2(3^{-1/4}))^2 + (\pm 3^{1/4})^2} = \sqrt{\frac{5}{3^{1/2}} + 3^{1/2}} = \sqrt{8}.
\]
Let \( f(x, y) = 4xy^2 - x^2y^2 - xy^3 \). Let \( D \) be the closed triangular region in the \( xy \)-plane with vertices \((0, 0)\), \((0, 6)\) and \((6, 0)\). Find the absolute maximum and minimum values of \( f \) on \( D \).

First we find the critical points of \( f \). The gradient is \( \nabla f = (4y^2 - 2xy^2 - y^3, 8xy - 2x^2y - 3xy^2) \). This gives us the system of equations

\[
4y^2 - 2xy^2 - y^3 = 0 \\
8xy - 2x^2y - 3xy^2 = 0.
\]

We can rewrite this after some factoring as

\[
y^2(4 - 2x - y) = 0 \\
x(8 - 2x - 3y) = 0.
\]

When one of the factors \( y^2 \) or \( xy \) is zero, the critical point will not be inside the domain – it will be on the boundary, or maybe even outside the domain entirely, so we do not need to worry about those here. (We will check the boundary later, anyway.) So we really only need to consider the system of equations

\[
4 - 2x - y = 0 \\
8 - 2x - 3y = 0.
\]

Subtracting the second equation from the first yields \( 4 - 2y = 0 \), so \( y = 2 \). Plug that back into either equation to obtain \( x = 1 \). So the only critical point inside the domain is \((1, 2)\). Notice that \( f(1, 2) = 4 \).

Next we check the boundary. There are three line segments that form the boundary of \( D \): \( x = 0 \), \( y = 0 \) and \( x = 6 - y \). On the first two pieces, \( f(x, 0) = 0 \) and \( f(0, y) = 0 \). On the last piece, we have

\[
f(6 - y, y) = 4(6 - y)y^2 - (6 - y)^2y^2 - (6 - y)y^3 \\
= 2y^3 - 12y.
\]

This segment corresponds to \( 0 \leq y \leq 6 \), so we use the methods of 1-variable calculus to find the absolute maximum and minimum there. We take an ordinary derivative of this expression with respect to \( y \) and set it equal to zero to obtain the equation

\[
6y^2 - 12 = 0,
\]

so \( y = \pm\sqrt{2} \). But we need to have \( 0 \leq y \leq 6 \) on this segment, so we only need to check \( y = \sqrt{2} \). In that case, we have

\[
f(6 - \sqrt{2}, \sqrt{2}) = 2(\sqrt{2})^2 - 12\sqrt{2} = 4 - 12\sqrt{2} < 0.
\]

That is to say, \( f(6\sqrt{2}, \sqrt{2}) = 4 - \sqrt{2} \). We have already checked the endpoints of this segment since they lie on the coordinate axes, where we know that \( f = 0 \).

Consequently, we see that the absolute maximum of \( f \) on the domain \( D \) is 4, and it occurs at the point \((1, 2)\), and the absolute minimum is \( 4 - 12\sqrt{2} \) which occurs at the point \((6 - \sqrt{2}, \sqrt{2})\).
Find the critical points of the function \( f(x, y) = 3xy - x^2y - xy^2 \) and classify them as local maxima, local minima or saddle points.

The gradient of \( f \) is \( \nabla f = (3y - 2xy - y^2, 3x - x^2 - 2xy) \). Setting this equal to the zero vector gives us the system of equations

\[
\begin{align*}
3y - 2xy - y^2 &= 0 \\
3x - x^2 - 2xy &= 0.
\end{align*}
\]

Factoring yields

\[
\begin{align*}
y(3 - 2x - y) &= 0 \\
x(3 - x - 2y) &= 0.
\end{align*}
\]

Since either factor of either equation can be zero, this really gives us 4 systems of equations:

1. \( y = 0 \) and \( x = 0 \)
2. \( y = 0 \) and \( 3 - x - 2y = 0 \)
3. \( 3 - 2x - y = 0 \) and \( x = 0 \)
4. \( 3 - 2x - y = 0 \) and \( 3 - x - 2y = 0 \).

Solving each of these systems gives us the following four solutions:

\((0, 0), \ (3, 0), \ (0, 3) \) and \((1, 1)\).

Now we classify these critical points. The second derivatives of \( f \) are

\[
\begin{align*}
f_{xx} &= -2y, \\
f_{xy} &= 3 - 2x - 2y, \\
f_{yy} &= -2x.
\end{align*}
\]

Thus the discriminator is

\[
D(x, y) = f_{xx}f_{yy} - (f_{xy})^2 = 4xy - (3 - 2x - 2y)^2.
\]

Therefore \( D(0, 0) = -9 < 0 \), so \((0, 0)\) is a saddle point; \( D(3, 0) = -9 < 0 \), so \((3, 0)\) is a saddle point; \( D(0, 3) = -9 < 0 \), so \((0, 3)\) is a saddle point; and \( D(1, 1) = 3 > 0 \) and \( f_{xx}(1, 1) = -2 < 0 \), so \((1, 1)\) is a local maximum.
Find the direction in which \( f(x, y, z) = ze^{x+y} \) increases most rapidly at the point \((0, 1, 2)\). What is the maximum rate of increase?

The gradient is \( \nabla f = (z e^{x+y}, z e^{x+y}, e^{x+y}) \). Hence \( \nabla f(0, 1, 2) = (2, 0, 1) \).

This is the direction in which \( f \) increases most rapidly (equivalently, it increases fastest in the direction of the unit vector \( \frac{\nabla f(0, 1, 2)}{|\nabla f(0, 1, 2)|} = \frac{(2, 0, 1)}{\sqrt{5}} \)).

The rate of change is \( |\nabla f(0, 1, 2)| = |(2, 0, 4)| = \sqrt{5} \).

If \( z = f(x^2 - y^2) \), where \( f \) is differentiable, show that \( y \frac{\partial z}{\partial x} + x \frac{\partial z}{\partial y} = 0 \).

We use the chain rule to calculate \( \frac{\partial z}{\partial x} = f'(x^2 - y^2) \frac{\partial}{\partial x} [x^2 - y^2] = f'(x^2 - y^2) 2x \) and \( \frac{\partial z}{\partial y} = f'(x^2 - y^2) \frac{\partial}{\partial y} [x^2 - y^2] = -f'(x^2 - y^2) 2x \).

Consequently, \( y \frac{\partial z}{\partial x} + x \frac{\partial z}{\partial y} = y(f'(x^2 - y^2) 2x) + x(-f'(x^2 - y^2) 2x) = 2xyf'(x^2 - y^2) - 2xyf'(x^2 - y^2) \)

\[ = 0. \]

Calculate \( \int \int \int_B x^2 + y^2 \, dV \), where \( B \) is the unit ball in \( \mathbb{R}^3 \): \( B = \{ (x, y, z); x^2 + y^2 + z^2 \leq 1 \} \).

We use spherical coordinates to calculate the triple integral:

\[
\int \int \int_B x^2 + y^2 \, dV = \int_0^\pi \int_0^1 \int_0^{2\pi} (\rho \sin(\phi) \cos(\theta))^2 + (\rho \sin(\phi) \sin(\theta))^2 \rho^2 \sin(\phi) \, d\theta \, d\rho \, d\phi \\
= \int_0^\pi \int_0^1 \int_0^{2\pi} \rho^4 \sin^3(\phi) \, d\theta \, d\rho \, d\phi \\
= 2\pi \int_0^\pi \int_0^1 \rho^4 \sin^3(\phi) \, d\rho \, d\phi \\
= \frac{2\pi}{5} \int_0^\pi \sin^3(\phi) \, d\phi \\
= \frac{2\pi}{5} \int_0^\pi (1 - \cos^2(\phi)) \sin(\phi) \, d\phi \\
= -\frac{2\pi}{5} \int_1^{-1} (1 - s^2) \, ds \quad (s = \cos(\phi), \ ds = -\sin(\phi) \, d\phi) \\
= -\frac{2\pi}{5} \left( s - \frac{s^3}{3} \right) \bigg|_1^1 = \frac{8\pi}{15}.
\]
Calculate $\int \int_S x^2 + y^2 \, dS$, where $S$ is the unit sphere in $\mathbb{R}^3$: $S = \{(x, y, z); x^2 + y^2 + z^2 = 1\}$.

We parametrize the sphere using spherical coordinates, keeping in mind that $\phi = 1$ everywhere on the sphere:

$$\vec{r}(\theta, \phi) = (\sin(\phi) \cos(\theta), \sin(\phi) \sin(\theta), \cos(\phi)).$$

The partial derivatives are

$$\vec{r}_\theta = (-\sin(\phi) \sin(\theta), \sin(\phi) \cos(\theta), 0)$$

and

$$\vec{r}_\phi = (\cos(\phi) \cos(\theta), \cos(\phi) \sin(\theta), -\sin(\phi)).$$

Therefore

$$\vec{r}_\theta \times \vec{r}_\phi = (-\sin^2(\phi) \cos(\theta), -\sin^2(\phi) \sin(\theta), -\sin(\phi) \cos(\phi)).$$

Using the pythagorean identity to simplify yields

$$|\vec{r}_\theta \times \vec{r}_\phi| = \sqrt{\sin^4(\phi) \cos^2(\theta) + \sin^4(\phi) \sin^2(\theta) + \sin^2(\phi) \cos^2(\phi)}$$

$$= \sqrt{\sin^4(\phi) + \sin^2(\phi) \cos^2(\phi)}$$

$$= \sqrt{\sin^2(\phi)}$$

$$= \sin(\phi).$$

In the last line, we used the fact that $0 \leq \phi \leq \pi$ implies $\sin(\phi) \geq 0$, so that we didn’t have to be concerned with absolute values. Now we can compute

$$\int \int_S x^2 + y^2 \, dS = \int_0^\pi \int_0^{2\pi} ((\sin(\phi) \cos(\theta))^2 + (\sin(\phi) \sin(\theta))^2) \sin(\phi) \, d\theta \, d\phi$$

$$= \int_0^\pi \int_0^{2\pi} \sin^3(\phi) \, d\theta \, d\phi$$

$$= 2\pi \int_0^\pi \sin^3(\phi) \, d\phi$$

$$= 2\pi \int_0^\pi (1 - \cos^2(\phi)) \sin(\phi) \, d\phi$$

$$= -2\pi \int_1^{-1} (1 - s^2) \, ds \quad (s = \cos(\phi), \, ds = -\sin(\phi) \, d\phi)$$

$$= -2\phi \left[ s - \frac{s^3}{3} \right]_{-1}^{1}$$

$$= \frac{8\pi}{3}.$$
Let $E$ be the region bounded by the paraboloids $z = 4 - x^2 - y^2$ and $z = x^2 + y^2$. Compute $\iiint_E x^2 \, dV$.

The paraboloid $z = 4 - x^2 - y^2$ opens downward, while $z = x^2 + y^2$ opens upward. The paraboloids intersect in a circle: $4 - x^2 - y^2 = x^2 + y^2$ implies $4 = 2x^2 + y^2$, so $2 = x^2 + y^2$; hence $z = 4 - 2 = 2$. That is to say, the intersection of the paraboloids is the circle $x^2 + y^2 = 2$ in the plane $z = 2$. If we “stand on top of the $z$-axis and look down” we see that the domain in the $xy$-plane is the circle centered at the origin with radius $\sqrt{2}$. Now if we use cylindrical coordinates for the integral, we get

\[
0 \leq r \leq \sqrt{2}, \quad 0 \leq \theta \leq 2\pi \quad \text{and} \quad r^2 \leq z \leq 4 - r^2.
\]

Therefore

\[
\iiint_E x^2 \, dV = \int_0^{2\pi} \int_0^{\sqrt{2}} \int_0^{4-r^2} (r \cos \theta)^2 r \, dz \, dr \, d\theta
\]

\[
= \int_0^{2\pi} \int_0^{\sqrt{2}} \int_0^{4-r^2} r^3 \cos^2 \theta \, dz \, dr \, d\theta
\]

\[
= \int_0^{2\pi} \int_0^{\sqrt{2}} rz^2 \cos^2 \theta \bigg|_{z=4-r^2} \, dr \, d\theta
\]

\[
= \int_0^{2\pi} \left( 4r^3 - 2r^5 \right) \cos^2 \theta \, dr \, d\theta
\]

\[
= \int_0^{2\pi} \left( 4 - \frac{8}{3} \right) \cos \theta \, d\theta
\]

\[
= \frac{16}{3} \int_0^{2\pi} \cos^2 \theta \, d\theta
\]

\[
= \frac{16}{3} \int_0^{2\pi} \frac{1 + \cos(2\theta)}{2} \, d\theta
\]

\[
= \frac{8}{3} \left( \theta + \frac{\sin(2\theta)}{2} \right) \bigg|_0^{2\pi}
\]

\[
= \frac{16\pi}{3}.
\]

Let $\vec{F}(x, y, z) = (2xy^3 + z^2)\hat{i} + (3x^2y^2 + 2yz)\hat{j} + (y^2 + 2xz)\hat{k}$. (a) Find a potential function $f$ for $\vec{F}$. (b) Compute $\int_C \vec{F} \cdot d\vec{r}$, where $C$ is the curve $\vec{r}(t) = \langle t, t^2, t^3 \rangle$, with $0 \leq t \leq 1$.

(a) The potential function needs to satisfy $\frac{\partial f}{\partial x} = 2xy^3 + z^2$, so we have

\[
f = x^2y^3 + xz^2 + C(y, z).
\]

Differentiating with respect to $y$ yields

\[
\frac{\partial f}{\partial y} = 3x^2y^2 + \frac{\partial C}{\partial y},
\]
but this must equal $3x^2y^2 + 2yz$, so we have $\frac{\partial C}{\partial y} = 2yz$; hence $C(y, z) = y^2z + D(z)$. Thus

$$f = x^2y^3 + xz^2 + y^2z + D(z),$$

and differentiating with respect to $z$ yields

$$\frac{\partial f}{\partial z} = 2xz + y^2 + D'(z).$$

We need to have $\frac{\partial f}{\partial z} = 2xz + y^2$, so $D'(z) = 0$. Thus we may take the potential function to be

$$f(x, y, z) = x^2y^3 + xz^2 + y^2z.$$

(b) The given path $C$ begins at $(0, 0, 0)$ and ends at $(1, 1, 1)$. Therefore, using the Fundamental Theorem for Line Integrals gives us

$$\int_C \vec{F} \cdot d\vec{r} = f(1, 1, 1) - f(0, 0, 0) = 1 + 1 + 0 - 0 - 0 - 0 = 3.$$

10 Compute $\oint_C (y^3 + \tan x) \, dx - (x^3 + \sin y) \, dy$, where $C$ is the positively oriented boundary of the region between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

We use Green’s Theorem with $P(x, y) = y^3 + \tan x$ and $Q(x, y) = -(x^3 + \sin y)$ and $D$ the region between these circles to get

$$\int_C P \, dx + Q \, dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA$$

$$= \iint_D (-3x^2 - 3y^2) \, dA$$

$$= -\int_1^2 \int_0^{2\pi} 3r^2 r \, d\theta \, dr$$

$$= -6\pi \int_1^2 r^3 \, dr$$

$$= \left. -\frac{3\pi r^4}{2} \right|_1 = \boxed{-\frac{45\pi}{2}}.$$

11 Show that there is no vector field $\vec{G}$ such that $\text{curl} \vec{G} = \langle 2x, 3yz, -xz^2 \rangle$.

Write $\vec{F} = \langle 2x, 3yz, -xz^2 \rangle$. Suppose that $\vec{F} = \text{curl} \vec{G}$. Then $\text{div} \vec{F} = \text{div} \text{curl} \vec{G} = 0$, because the divergence of the curl of any vector field is the zero vector field. However, for this vector field $\vec{F}$, we have

$$\text{div} \vec{F} = \nabla \cdot \langle 2x, 3yz, -xz^2 \rangle = \frac{\partial}{\partial x}[2x] + \frac{\partial}{\partial y}[3yz] + \frac{\partial}{\partial z}[-xz^2] = 2 + 3z - 2xz.$$

This is not the zero vector field, so $\vec{F}$ cannot be the curl of $\vec{G}$. 
Let $S$ be the surface of the bottomless box with corners $(±1, ±1, ±1)$ and with outward orientation (that is to say, choose the outward orientation as if the box was closed). Let $\vec{F} = \langle x+y^2, x+z^3, z \rangle$. (a) Find $\int_S \vec{F} \cdot d\vec{S}$. (b) Find $\int \int (\text{curl} \ \vec{F}) \cdot d\vec{S}$.

(a) Let $S_2$ be the bottom of the box – that is to say, $S_2$ is the square $-1 \leq x \leq 1, -1 \leq y \leq 1$ in the plane $z = -1$. Let $S_2$ have the downward orientation. Then together, $S$ and $S_2$ are the outward-oriented boundary of the box $E$ given by $-1 \leq x \leq 1, -1 \leq y \leq 1$ and $-1 \leq z \leq 1$. According to the Divergence Theorem,

$$\int\int_S \vec{F} \cdot d\vec{S} + \int\int_{S_2} \vec{F} \cdot d\vec{S} = \int\int\int_E \text{div} \ F \ dV$$

$$= \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} \nabla \cdot (x+y^2, x+z^3, z) \ dV$$

$$= \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} 2dV$$

$$= 2(\text{Volume of } E)$$

$$= 16.$$  

To finish solving for the integral over $S$, we need to know the value of the integral over $S_2$. We can compute that directly with the parametrization

$$\vec{r}(x, y) = \langle x, y, -1 \rangle.$$  

We calculate

$$\vec{r}_x(x, y) = \langle 1, 0, 0 \rangle$$

and

$$\vec{r}_y(x, y) = \langle 0, 1, 0 \rangle.$$  

So

$$\vec{r}_x(x, y) \times \vec{r}_y(x, y) = \langle 0, 0, 1 \rangle.$$  

This has upward orientation, but the normal vector we need has downward orientation:

$$\vec{r}_y(x, y) \times \vec{r}_x(x, y) = \langle 0, 0, -1 \rangle.$$  

Therefore

$$\int\int_{S_2} \vec{F} \cdot d\vec{S} = \int_{-1}^{1} \int_{-1}^{1} (x+y^2, x+1, 1) \cdot (0, 0, -1) \ dV$$

$$= \int_{-1}^{1} \int_{-1}^{1} -1 \ dV$$

$$= -(\text{Area of } [-1, 1] \times [-1, 1])$$

$$= -2.$$  

Hence

$$\int\int_S \vec{F} \cdot d\vec{S} - 2 = 16,$$
so
\[ \iint_S \vec{F} \cdot d\vec{S} = 18. \]

(b) Let \( C \) be the positively-oriented boundary of \( C \): it is the square curve from \((-1, -1, -1)\) to \((1, -1, -1)\) to \((1, 1, -1)\) to \((-1, 1, -1)\) and back to the starting corner. The according to Stoke’s Theorem, we get
\[ \iint_S \text{curl} \vec{F} \cdot d\vec{S} = \oint_C \vec{F} \cdot d\vec{r}. \]

But notice that this is the same value that we get for \( \iint_{S_2} \text{curl} \vec{F} \cdot d\vec{S} \), where \( S \) is the square above but with \textit{upward} orientation:
\[ \iint_{S_2} \text{curl} \vec{F} \cdot d\vec{S} = \oint_C \vec{F} \cdot d\vec{r}. \]

Thus
\[ \iint_S \text{curl} \vec{F} \cdot d\vec{S} = \iint_{S_2} \text{curl} \vec{F} \cdot d\vec{S}. \]

The point is that the surface integral over \( S_2 \) is much easier to calculate. First we need the curl:
\[ \text{curl} \vec{F} = \nabla \times \vec{F} = (3z^2, 0, 1). \]

Therefore
\[
\iint_{S_2} \text{curl} \vec{F} \cdot d\vec{S} = \int_{-1}^{1} \int_{-1}^{1} (3, 0, 1) \cdot (0, 0, 1) \, dx \, dy \\
= \int_{-1}^{1} \int_{-1}^{1} 1 \, dx \, dy \\
= -4.
\]

That is to say,
\[ \iint_S \text{curl} \vec{F} \cdot d\vec{S} = -4. \]