1. \[ V = \int \int_D (x^2 + y) \, dA \]
   \[ = \int_0^2 \left[ \int_0^y \left( \frac{x^3}{3} + xy \right) \, dx \right] \, dy \]
   \[ = \int_0^2 \left[ \frac{x^4}{3} + \frac{xy^2}{2} \right]_0^y \, dy \]
   \[ = \int_0^2 \left( \frac{y^4}{12} + \frac{y^3}{3} \right) \, dy \]
   \[ = \left[ \frac{y^5}{60} + \frac{y^4}{12} \right]_0^2 \]
   \[ = \frac{16}{15} + \frac{8}{3} \]
   \[ = \frac{4}{3} \]

2. \[ \int_0^{\sqrt{\pi}} \int_0^{\sqrt{\pi}} 2 \cos(y^2) \, dy \, dx \]
   \[ = \int_0^{\sqrt{\pi}} \left[ \int_0^{\sqrt{\pi}} 2 \cos(y^2) \, dx \right] \, dy \]
   \[ = \int_0^{\sqrt{\pi}} \left[ 2 \cos(y^2) \right]_0^{\sqrt{\pi}} \, dy \]
   \[ = \int_0^{\sqrt{\pi}} 2 \cos(y^2) \, dy \]
   \[ = \int_0^{\sqrt{\pi}} \cos(y^2) \, dy \]
\[= \int_0^{\sqrt{\pi}} 2y \cos(y^2) \, dy\]
\[= \int_{u=\pi} \cos(u) \, du\]
\[= \sin(\pi) - \sin(0)\]
\[= 0\]
Thus
\[
\bar{x} = \frac{1}{M} \int \int x(x,y) \, dA = \frac{1}{9} \int \int r \cos \theta \sin \theta \, r \, dr \, d\theta
\]
\[
= \frac{1}{9} \int_{0}^{\frac{\pi}{2}} \int_{0}^{\frac{\pi}{2}} r^3 \sin \theta \cos \theta \, r \, dr \, d\theta = \frac{1}{9} \int_{0}^{\frac{\pi}{2}} \frac{81}{4} \sin \theta \cos \theta \, d\theta
\]
\[
= \frac{9}{4} \int_{0}^{\frac{\pi}{2}} \sin \theta \cos \theta \, d\theta = \frac{9}{4} \int_{0}^{u=1} u \, du
\]
\[
= \frac{9}{8} u^2 \bigg|_{u=0}^{u=1} = \frac{9}{8}
\]
and
\[
\bar{y} = \frac{1}{M} \int \int y(x,y) \, dA = \frac{1}{9} \int \int p \sin \theta \cos \theta \, r \, dr \, d\theta
\]
\[
= \frac{1}{9} \int_{0}^{\frac{\pi}{2}} \int_{0}^{\frac{\pi}{2}} r^4 \sin^3 \theta \, d\theta = \frac{1}{9} \int_{0}^{\frac{\pi}{2}} \frac{81}{4} \sin^3 \theta \, d\theta
\]
\[
= \frac{9}{4} \int_{0}^{\frac{\pi}{2}} \sin^3 \theta \, d\theta = \frac{9}{4} \int_{0}^{\frac{\pi}{2}} \frac{1-\cos(2\theta)}{2} \, d\theta = \frac{9}{8} \int_{0}^{\frac{\pi}{2}} \cos \theta \, d\theta
\]
\[
= \frac{9}{8} \left( \frac{\pi}{2} \right) \cdot \frac{9}{16} \cos u \, du
\]
\[
= \frac{9\pi}{16} - \frac{9}{16} \sin u \bigg|_{u=0}^{u=\frac{\pi}{2}} = \frac{9\pi}{16}
\]
So
\[
(\bar{x}, \bar{y}) = \left( \frac{9}{4}, \frac{9\pi}{16} \right)
\]
\[ \text{Surface area} = \iint_R \sqrt{1 + \left( \frac{\partial z}{\partial x} \right)^2 + \left( \frac{\partial z}{\partial y} \right)^2} \, dA \]
\[ = \iint_R \sqrt{1 + (2y)^2 + (2y)^2} \, dA \]
\[ = \iint_R \sqrt{1 + 4r^2} \, dA \]
\[ = \int_0^{2\pi} \int_0^1 \sqrt{1 + 4r^2} \, r \, dr \, d\theta \]
\[ = \frac{1}{8} \int_0^{2\pi} \int_0^5 \sqrt{u} \, du \, d\theta \]
\[ = \frac{1}{8} \int_0^{2\pi} \left[ \frac{2u^{3/2}}{3} \right]_0^5 \, d\theta \]
\[ = \frac{1}{12} \left( 5^{3/2} - 1 \right) \int_0^{2\pi} d\theta \]
\[ = \frac{(5^{3/2} - 1) \pi}{6} \]
For the first square root to be defined, we need to have

$$9 - x^2 - y^2 \geq 0$$

So $x^2 + y^2 \leq 9$.

This is the region inside a circle centered at the origin with radius 3, including the boundary.

For the second square root to be defined, we need to have

$$x^2 + y^2 - 4 \geq 0$$

So $x^2 + y^2 \geq 4$.

This is the region outside a circle of radius 2, centered at the origin, including the boundary.

So the domain is

$$4 \leq x^2 + y^2 \leq 9,$$

which is illustrated as: 

![Diagram of a circle with radius 3 and 4, centered at the origin, including the boundary.]
\[ f(x, y) = \sqrt{9x^2 + 4y^2}. \]

\[ \sqrt{9x^2 + 4y^2} = k \]

\[ 9x^2 + 4y^2 = k^2 \]

These are ellipses centered at the origin with x-intercepts of \( x = \pm \frac{k}{3} \) and y-intercepts of \( y = \pm \frac{k}{2} \).
\[ f(x, y) = xy \sin(xy) - x^2 y^3 \]

\[ f_x = \sin(xy) + xy \cos(xy) - 2xy^3 \]

\[ f_y = x^2 \cos(xy) - 3x^2 y^2 \]

\[ f_x(1, \pi) = \sin(\pi) + \pi \cos(\pi) - 2\pi^3 \]
\[ = 2\pi^3 - \pi^3 \]

\[ f_y(1, \pi) = \cos(\pi) - 3\pi^2 \]
\[ = -1 - 3\pi^2. \]

So the linearization is

\[ L(x, y) = f(1, \pi) + f_x(1, \pi)(x-1) + f_y(1, \pi)(y-\pi) \]

\[ L(x, y) = -\pi^2 + (2\pi^3 - \pi)(x-1) + (-1 - 3\pi^2)(y-\pi) \]

\[ f(x, y) = xe^{-2y} \]

\[ f_x = e^{-2y} \]

\[ f_y = -2xe^{-2y} \]

\[ f_{xx} = 0 \]

\[ f_{xy} = -2e^{-2y} \]

\[ f_{yy} = 4xe^{-2y} \]

\[ f_{yx} = -2e^{-2y} \]
\[ f = \cos(x + 2y) \]
\[ \frac{df}{dx} = -\sin(x + 2y) \]
\[ \frac{df}{dy} = -2\sin(x + 2y) \]

When \( t = 1, \ x = 1 \) and \( y = 0 \), so
\[ \frac{df}{dx} \bigg|_{t=1} = -\sin(1) \]

and
\[ \frac{df}{dy} \bigg|_{t=1} = -2\sin(1) \]

Therefore
\[
\sqrt{\frac{df}{dt}} = \frac{df}{dx} \frac{dx}{dt} + \frac{df}{dy} \frac{dy}{dt}
\]
\[ = (-\sin(1))(2) + (-2\sin(1))(2) \]
\[ = -6\sin(1) \]
10. \( \bar{r}(u,v) = (u,v, u+v, u-v) \), \( 0 \leq u \leq 1, \ 0 \leq v \leq 2. \)

\( \bar{r}_u = (v, 1, 1) \)

\( \bar{r}_v = (u, 1, -1) \)

\( \bar{r}_u \times \bar{r}_v = \det \begin{bmatrix} i & j & k \\ v & 1 & 1 \\ u & 1 & -1 \end{bmatrix} = -2i + (u+v)j + (v-u)k \)

\[ = (-2, u+v, v-u). \]

(a) Therefore, when \( u = 2 \) and \( v = 1 \), we have

\[ (\bar{r}_u \times \bar{r}_v)(2,1) = (-2, 3, -1). \]

This is a normal vector for the tangent plane through the point with position vector

\[ \bar{r}(2,1) = (2, 3, 1). \]

So the equation for the plane is (using the formula \( \bar{n} \cdot (x-x_0, y-y_0, z-z_0) = 0 \)):

\[ <-2, 3, -1> \cdot (x-2, y-3, z-1) = 0 \]

\[ \Rightarrow -2(x-2) + 3(y-3) - (z-1) = 0. \]

(Or, this can be written in the form

\[ -2x + 3y - z = 4. \])
(b) \[
\text{surface area} = \iiint_0^2 |\vec{r}_u \times \vec{r}_v| \, du \, dv
\]

we already calculated:

\[
\vec{r}_u \times \vec{r}_v = \langle -2, u+v, v-u \rangle
\]

so

\[
|\vec{r}_u \times \vec{r}_v| = |\langle -2, u+v, v-u \rangle|
\]

\[
= \sqrt{(-2)^2 + (u+v)^2 + (v-u)^2}
\]

\[
= \sqrt{4 + u^2 + 2uv + v^2 + v^2 - 2uv + u^2}
\]

\[
= \sqrt{4 + 2u^2 + 2v^2}
\]

Thus

\[
\text{surface area} = \iiint_0^2 \sqrt{4 + 2u^2 + 2v^2} \, du \, dv
\]

\[
\begin{bmatrix}
\begin{array}{c}
8
\end{array}
\end{bmatrix}
\]

\[
f(x,y) = x^2 + 2y
\]

\[
f_x = 2x
\]

\[
f_y = 2
\]

\[
\nabla f(2,3) = \langle 4, 2 \rangle
\]

\[
\nabla f(2,3) = \langle \frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}} \rangle \cdot \langle 4, 2 \rangle = \frac{6}{\sqrt{5}}
\]
\[ f(x, y) = (x-1)^2(y-2)^2 \]

\[ f_x = 2(x-1)(y-2)^2 \]

\[ f_y = 2(x-1)^2(y-2) \]

\[ \nabla f = \langle 2(x-1)(y-2)^2, 2(x-1)^2(y-2) \rangle = 0 \quad \text{when} \]

\[ \begin{cases} 
2(x-1)(y-2)^2 = 0 \\
2(x-1)^2(y-2) = 0 
\end{cases} \]

This system is solved by \( x=1 \) or \( y=2 \).

That is to say, at any point where \( x=1 \), \( f \) has a critical point, and it has a critical point everywhere \( y=2 \). But also,

\[ f(1,y) = 0 \quad \text{for all } y \]

and

\[ f(x,2) = 0 \quad \text{for all } x. \]

We still need to check the boundary of the domain:
**Bottom Edge**

\[ f(x,0) = 4(x-1)^2 \quad \text{for} \quad 0 \leq x \leq 2. \]

\[ \frac{d}{dx} \left[ f(x-1)^2 \right] = 8(x-1) = 0 \quad \text{when} \quad x=1 \]

\[ f(1,0) = 0, \quad f(0,0) = 4, \quad f(2,0) = 4 \]

Critical point on edge
endpoints of edge.

**Top Edge**

\[ f(x,3) = (x-1)^2 \]

\[ \frac{d}{dx} \left[ (x-1)^2 \right] = 2(x-1) = 0 \quad \text{when} \quad x=1 \]

\[ f(1,3) = 0, \quad f(0,3) = 1, \quad f(2,3) = 1. \]

**Left Edge**

\[ f(0,y) = (y-2)^2 \]

\[ \frac{d}{dy} \left[ (y-2)^2 \right] = 2(y-2) = 0 \quad \text{when} \quad y=2 \]

\[ f(0,2) = 0, \quad f(0,0) = 4, \quad f(0,3) = 1. \]
\[ f(2, 2) = (1 - 2)^2 \]
\[ \frac{d}{dx} [(4 - x)^2] = 2(4 - x) = 0 \quad \text{when } x = 2 \]
\[ f(2, 2) = 0, \quad f(2, 0) = 4, \quad f(2, 3) = 1. \]

Comparing all these values at critical points and endpoints, we see that the

Absolute Maximum is 4

and the

Absolute Minimum is 0.

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\[ f(x, y, z) = x + 2y - 3z \]
\[ g(x, y, z) = x^2 + y^2 + z^2 \quad \text{constraint: } g = 1. \]

\[ \nabla f = \langle 1, 2, -3 \rangle \]
\[ \nabla g = \langle 2x, 2y, 2z \rangle. \]

\[ \nabla f = \lambda \nabla g \quad \text{gives us} \]
\[ \begin{cases} 
1 = 2\lambda x \\
2 = 2\lambda y \\
-3 = 2\lambda z 
\end{cases} \]
So \[ x = \frac{1}{2\lambda}, \]
\[ y = \frac{1}{\lambda}, \]
\[ z = \frac{-3}{2\lambda}. \]

In particular, \( y = 2x \) and \( z = -3x \).

If we substitute these into the constraint equation \( x^2 + y^2 + z^2 = 1 \), we get

\[ x^2 + (2x)^2 + (-3x)^2 = 1 \]

\[ \Rightarrow \quad x^2 + 4x^2 + 9x^2 = 1 \]

\[ \Rightarrow \quad 14x^2 = 1 \]

\[ \Rightarrow \quad x = \pm \sqrt{\frac{1}{14}} = \pm \frac{1}{\sqrt{14}}. \]

If \( x = \frac{1}{\sqrt{14}} \), we get \( y = \frac{2}{\sqrt{14}} \) and \( z = \frac{-3}{\sqrt{14}} \), so

\[ f\left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{-3}{\sqrt{14}}\right) = \frac{1}{\sqrt{14}} + 2\left(\frac{2}{\sqrt{14}}\right) - 3\left(\frac{-3}{\sqrt{14}}\right) = \frac{14}{\sqrt{14}} = \frac{1}{\sqrt{14}}. \]

If \( x = -\frac{1}{\sqrt{14}} \), we get \( y = \frac{-2}{\sqrt{14}} \) and \( z = \frac{3}{\sqrt{14}} \), so

\[ f\left(\frac{-1}{\sqrt{14}}, \frac{-2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right) = \frac{-1}{\sqrt{14}} + 2\left(\frac{-2}{\sqrt{14}}\right) - 3\left(\frac{3}{\sqrt{14}}\right) = \frac{-14}{\sqrt{14}} = \frac{-1}{\sqrt{14}}. \]

So the maximum value is \( \boxed{\frac{1}{\sqrt{14}}} \).