\[ \begin{cases} \frac{dy}{dx} = xy^3 \\ y(1) = -3 \end{cases} \]

**Use separation of variables:**

\[ \int \frac{dy}{y^3} = \int x \, dx \]

So \[ \frac{-1}{2y^2} = \frac{x^2}{2} + C \]

Hence \[ y^2 = \frac{-1}{x^2 + C} \]

So \[ y = \pm \sqrt{\frac{-1}{x^2 + C}} \]

Since \( y(1) = -3 \), we need the negative sign:

\[ y = -\sqrt{\frac{-1}{x^2 + C}} \]

And we can solve for \( C \):

\[ -3 = -\sqrt{\frac{1}{-1 + C}} \]

\[ \Rightarrow 9 = \frac{1}{-1 + C} \]

\[ \Rightarrow \frac{1}{9} = -1 + C \]

\[ \Rightarrow \frac{10}{9} = C \]

So

\[ y = -\sqrt{\frac{-1}{x^2 + \frac{10}{9}}} \]

Note: we could also have solved this as a Bernoulli equation.
\[ \begin{align*}
\frac{dy}{dx} &= e^x \cos^2(y) \\
y(0) &= 0
\end{align*} \]

Use separation of variables:

\[ \int \sec^2(y) dy = \int e^x dx \]

so

\[ \tan(y) = e^x + C \]

Since \( y(0) = 0 \), we get

\[ \tan(0) = e^0 + C \]

so

\[ C = -1 \]

Hence \( C = -1 \). Therefore

\[ \tan(y) = e^x - 1 \]

and consequently

\[ y = \tan^{-1}(e^x - 1) + k \pi \]

Since \( y(0) = 0 \), we get \( k = 0 \). So

\[ y = \tan^{-1}(e^x - 1) \]

Therefore

\[ \lim_{x \to \infty} \tan^{-1}(e^x - 1) = \frac{\pi}{2} \]
\[ \frac{dy}{dx} = y^2 + y = y(y+1) \]

From the phase portrait, we see that

\[ \lim_{x \to \infty} y(x) = \begin{cases} 
\infty & \text{if } y(0) > 0 \\
0 & \text{if } y(0) = 0 \\
-1 & \text{if } y(0) < 0 
\end{cases} \]
Let $E(t)$ be the volume of ethanol in the tank after $t$ minutes. Then

$$E(0) = 0 \quad \text{and} \quad \frac{dE}{dt} = 3 - \frac{3E}{100}$$

We will solve this using an integrating factor:

$$\frac{dE}{dt} + \frac{3}{100} E = 3$$

the integrating factor is $e^{\frac{3t}{100}}$, so

$$e^{\frac{3t}{100}} \frac{dE}{dt} + e^{\frac{3t}{100}} \frac{3}{100} E = 3 e^{\frac{3t}{100}}$$

or

$$\frac{d}{dt} \left[ e^{\frac{3t}{100}} E \right] = 3 e^{\frac{3t}{100}}$$

Integrate both sides with respect to $t$:

$$e^{\frac{3t}{100}} E = 100 e^{\frac{3t}{100}} + C$$

Using $E(0) = 0$ gives us

$$0 = 100 + C$$

so $C = -100$. Therefore $e^{\frac{3t}{100}} E = 100 e^{\frac{3t}{100}} - 100$, and consequently

$$E = 100 - 100 e^{\frac{3t}{100}}$$

After 30 minutes, the volume will be $E(30) \approx 59$ gallons. Therefore, the percentage of liquid in the tank that is ethanol will be $59\%$. 
\[ \frac{dv}{dt} = -9.8 + kv \]

This is autonomous, so we can analyze it with a phase portrait:

![Phase Portrait Diagram]

Recall that \( k \) is negative.

The limit is \( \lim_{t \to \infty} v(t) = \frac{9.8}{k} \).

Since we're told this limit should equal \(-100\),
we get

\[ \frac{9.8}{k} = -100 \]

So \( k = -0.098 \).

Note that we could also have answered this by solving the differential equation instead of by using the phase portrait.
\[ \frac{dp}{dt} = kp(4-p) \quad p \text{ in millions} \]

\[ p(0) = 0.03 \quad \text{and} \quad p(0) = 1 \]

So \[ 0.03 = k(1)(4-1) = 3k \]

Thus \[ k = 0.01. \]

Therefore we need to solve the initial value problem

\[ \begin{cases} \frac{dp}{dt} = 0.01 p(4-p) \\ p(0) = 1 \end{cases} \]

Let's solve this using a substitution. Write

\[ \frac{dp}{dt} - 0.04p = -0.01p^2 \]

Note: we could have

\[ \text{separated variables instead.} \]

Set \[ u = \frac{1}{p}, \text{ so} \]

\[ \frac{du}{dt} = -\frac{1}{p^2} \frac{dp}{dt} = \frac{1}{p^2} \left[ 0.04p - 0.01p^2 \right] = -0.04 \frac{1}{p} + 0.01 = -0.04u + 0.01 \]

Hence

\[ \frac{du}{dt} + 0.64u = 0.01 \]
Now use the integrating factor $e^{0.04t}$.

\[
\frac{d}{dt} [e^{0.04t} u] = 0.01e^{0.04t}
\]

Hence

\[ e^{0.04t} u = \frac{1}{4} e^{0.04t} + C \]

So

\[ u = \frac{1}{4} + Ce^{-0.04t} \]

Therefore

\[ p = \frac{1}{\frac{1}{4} + Ce^{-0.04t}} \]

Use $p(0) = 1$ to get

\[ 1 = \frac{1}{\frac{1}{4} + C} \]

\[ \Rightarrow 1 = \frac{1}{\frac{1}{4} + C} \]

\[ \Rightarrow C = \frac{3}{4} \]

So

\[ P(t) = \frac{1}{\frac{1}{4} + \frac{3}{4}e^{-0.04t}} \]

We want to know when $P(t) = (0.90)(4) = 3.6$, so we solve

\[ 3.6 = \frac{1}{\frac{1}{4} + \frac{3}{4}e^{-0.04t}} \]

From which we get $t \approx 82$ years.
\[
\begin{cases}
\frac{dy}{dx} = (x+y)^2 \\
y(0) = 1
\end{cases}
\]

Let's use the substitution \( u = x + y \), so that
\[
\frac{du}{dx} = 1 + \frac{dy}{dx} = 1 + (x+y)^2 = 1 + u^2.
\]

Now let's separate variables:
\[
\int \frac{du}{1+u^2} = \int dx
\]

So
\[\tan^{-1}(u) = x + C.\]

Since \( y(0) = 1 \), \( u(0) = 0 + 1 = 1 \), and therefore
\[\tan^{-1}(1) = 0 + C\]

So
\[C = \tan^{-1}(1) = \frac{\pi}{4}.
\]

Consequently
\[\tan^{-1}(u) = x + \frac{\pi}{4}\]

So
\[u = \tan\left(x + \frac{\pi}{4}\right).
\]
8

(a) \[4y'' + y = 0\]

Characteristic equation: \[4r^2 + 1 = 0\]
\[r^2 = -\frac{1}{4}\]
\[r = \pm \frac{1}{2}i\]

So \[y(t) = A\sin\left(\frac{t}{2}\right) + B\cos\left(\frac{t}{2}\right)\]

(b) \[2x'' - 5x' = 3x\]
\[2x'' - 5x' - 3x = 0\]

Characteristic equation: \[2r^2 - 5r - 3 = 0\]
\[r = \frac{5 \pm \sqrt{25 - 4(-3)(2)}}{4}\]
\[r = \frac{5 \pm \sqrt{25 - 24}}{4}\]
\[r = \frac{5 \pm 1}{4}\]
\[r = \frac{5 \pm 7}{4}\]

So \[r = 3\] or \[r = -\frac{1}{2}\].

Thus \[x(t) = Ae^{3t} + Be^{-\frac{t}{2}}\]
(c) $\ddot{w} + 6\dot{w} + 9w = 0$

Characteristic equation: $r^2 + 6r + 9 = 0$

$(r+3)^2 = 0$

So $r = -3$ is a double root.

Thus

$y(t) = Ae^{-3t} + Be^{-3t}$

(d) $\dddot{V} - 4\ddot{V} + 13\dot{V} = 0$

Characteristic equation: $r^2 - 4r + 13 = 0$

$r = \frac{4 \pm \sqrt{16 - 4(13)}}{2}$

$= \frac{4 \pm \sqrt{-36}}{2}$

$= \frac{4 \pm 6i}{2}$

$= 1 \pm 3i$

So

$V(t) = Ae^{1t} \sin(3t) + Be^{1t} \cos(3t)$
9 Suppose \( A_1^2 + Br + c = 0 \). Let \( y = e^{rt} \).

Then \( y' = re^{rt} \) and \( y'' = r^2 e^{rt} \), so that

\[
AY'' + By' + cy = A(r^2 e^{rt}) + B(re^{rt}) + C(e^{rt})
\]

\[
= e^{rt} [A_1^2 + Br + C]
\]

\[
= e^{rt} \cdot 0
\]

\[
= 0.
\]

10 Let \( y_1 = e^{3t} \) and \( y_2(t) = u(t)e^{3t} \). Then

\[
y''_2 - 6y'_2 + 9y_2 = (u''(t)e^{2t} + 6u'(t)e^{3t} + 9u(t)e^{3t})
\]

\[
- 6(u'(t)e^{2t} + 3u(t)e^{3t})
\]

\[
+ 9(u(t)e^{3t})
\]

So we want to solve \( u''(t)e^{3t} = 0 \). Divide by \( e^{3t} \) to get \( u''(t) = 0 \). This implies \( u(t) = At + D \).

So \( y_2(t) = (At + D)e^{3t} \). Any function of this form is a solution of \( y'' - 6y' + 9y = 0 \).