Step Functions

In this worksheet, you will use the Laplace Transform of a step function to solve differential equations with piecewise-continuous driving functions.

Consider the function $f(t)$ given by

$$f(t) = \begin{cases} 
0 & \text{for } 0 \leq t \leq 1 \\
1 & \text{for } 1 \leq t \leq 3 \\
0 & \text{for } t \geq 3 
\end{cases}$$

(a) Write $f(t)$ as a difference of two step functions. (You can ignore what happens at the endpoints of subintervals). Then use this representation to write down the Laplace Transform of $f(t)$.

(b) Solve the initial-value problem $y'' = f(t), \ y(0) = 0, \ y'(0) = 0$ using the Laplace Transform. (You should have step functions in your answer.)
(c) Write your answer to (b) as a piecewise-defined function, without the notation of step functions. Then sketch your solution $y(t)$ on the interval $0 \leq t \leq 5$.

\[ g(t) = \begin{cases} 
  t & \text{for } 0 \leq t \leq 1 \\
  1 & \text{for } t \geq 1 
\end{cases} \]

2 Use the Laplace Transform with step functions to solve the initial-value problem:

\[ y'' = g(t), \quad y(0) = 0, \quad y'(0) = 0, \]

where

\[ g(t) = \begin{cases} 
  t & \text{for } 0 \leq t \leq 1 \\
  1 & \text{for } t \geq 1 
\end{cases} \]