(a) \[ 4y'' + y = 0 \]

Character equation:
\[ 4r^2 + 1 = 0 \]
\[ r^2 = -\frac{1}{4} \]
\[ r = \pm \frac{1}{2}i \]

So \[ y(t) = A\sin\left(\frac{t}{2}\right) + B\cos\left(\frac{t}{2}\right) \]

(b) \[ 2x'' - 5x' = 3x \]
\[ 2x'' - 5x' - 3x = 0 \]

Character equation:
\[ 2r^2 - 5r - 3 = 0 \]
\[ r = \frac{5 \pm \sqrt{25 - 4(-3)(2)}}{4} \]
\[ = \frac{5 \pm \sqrt{25 - 24}}{4} \]
\[ = \frac{5 \pm \sqrt{1}}{4} \]
\[ = \frac{5 \pm 1}{4} \]

So \[ r = 3 \quad \text{or} \quad r = -\frac{1}{2} \]

Thus \[ y(t) = Ae^{3t} + Be^{-\frac{t}{2}} \]
(c) \[ \ddot{w} + 6w + 9w = 0 \]

Characteristic equation: 
\[ r^2 + 6r + 9 = 0 \]
\[ (r + 3)^2 = 0 \]
So \( r = -3 \) is a double root.

Thus
\[ w(t) = Ae^{-3t} + Bte^{-3t} \]

(d) \[ \dddot{v} - 4\dot{v} + 13v = 0 \]

Characteristic equation: 
\[ r^2 - 4r + 13 = 0 \]
\[ r = \frac{4 \pm \sqrt{16 - 4(13)}}{2} \]
\[ = \frac{4 \pm \sqrt{-36}}{2} \]
\[ = \frac{4 \pm 6i}{2} \]
\[ = 2 \pm 3i \]

So
\[ v(t) = Ae^{2t}\sin(3t) + Be^{2t}\cos(3t) \]
(a) \[
\begin{aligned}
\begin{cases}
  y'' - 10y' + 25y = 0 \\
  y(0) = 1 \\
  y(1) = 1
\end{cases}
\end{aligned}
\]
Characteristic equation: \[
\begin{aligned}
r^2 - 10r + 25 &= 0 \\
(r - 5)^2 &= 0
\end{aligned}
\]
so \( r = 5 \) is a double root.

Hence
\[
\begin{aligned}
y(t) &= Ae^{5t} + Bte^{5t} \\
y(0) = 1 &\implies A = 1 \\
\text{so } y(t) &= e^{5t} + Bte^{5t} \\
y(1) = 1 &\implies e^5 + Be^5 = 1 \\
&\implies (B+1)e^5 = 1 \\
&\implies B + 1 = e^{-5} \\
&\implies B = e^{-5} - 1
\end{aligned}
\]
so
\[
\begin{aligned}
y(t) &= e^{5t} + (e^{-5} - 1)te^{5t}
\end{aligned}
\]
(b) \[
\begin{align*}
\ddot{x} + 4x &= 0 \\
x(0) &= 3 \\
x\left(\frac{\pi}{4}\right) &= 4
\end{align*}
\]

Characteristic equation: \( r^2 + 4 = 0 \)
\[ \Rightarrow r = \pm 2i \]

So
\[
x(t) = A \sin(2t) + B \cos(2t)
\]

\[ x(0) = 3 \Rightarrow B = 3 \]
\[ x\left(\frac{\pi}{4}\right) = 4 \Rightarrow A = 4 \]

So
\[
x(t) = 4 \sin(2t) + 3 \cos(2t)
\]
\[ (a) \begin{cases} y'' + 4y = 3\sin(x) \\ y(0) = 0 \\ y'(0) = 0 \end{cases} \]

Characteristic equation: \( r^2 + 4 = 0 \) \( \Rightarrow r = \pm 2i \)

\( \Rightarrow \) solution of homogeneous problem is

\[ y_c(x) = A\sin(2x) + B\cos(2x). \]

Since \( 3\sin(x) \) is not a solution of the homogeneous problem, our guess for a particular solution is

\[ y_p(x) = C\sin(x) + D\cos(x) \]

\( \Rightarrow \)

\[ y_p' = C\cos(x) - D\sin(x) \]

\( \Rightarrow \)

\[ y_p'' = -C\sin(x) - D\cos(x) \]

\( \Rightarrow \)

\[ y_p'' + 4y = (-C\sin(x) - D\cos(x)) + 4(C\sin(x) + D\cos(x)) = 3C\sin(x) + 3D\cos(x) \]

This must equal \( 3\sin(x) \), so we get \( C = 1, D = 0 \).

Hence \( y_p(x) = \sin(x) \).

Hence

\[ y(x) = \sin(x) + A\sin(2x) + B\cos(2x) \]

\( y(0) = 0 \) \( \Rightarrow \) \( B = 0 \)

\( y'(0) = 0 \) \( \Rightarrow \) \( 1 + 2A = 0 \) \( \Rightarrow A = \frac{1}{2} \)

So

\[ y(x) = \sin(x) + \frac{1}{2}\sin(2x) \]
(b) The characteristic equation and homogeneous solution are the same as in part (a). This time, since $3 \sin(2x)$ is a solution of the homogeneous problem, our guess for a particular nonhomogeneous solution will be

$$Y_p(x) = Ax \sin(2x) + Bx \cos(2x).$$

$$\Rightarrow Y_p'(x) = Asm(2x) + 2A \cos(2x) + B \cos(2x) - 2B \sin(2x).$$

$$\Rightarrow Y_p''(x) = 2A \cos(2x) + 2A \cos(2x) - 4A \sin(2x) - 2B \sin(2x) - 2B \sin(2x) - 4B \cos(2x) = 4A \cos(2x) - 4A \sin(2x) - 4B \sin(2x) - 4B \cos(2x)$$

So

$$Y_p'' + 4Y = 4A \cos(2x) - 4B \sin(2x).$$

Since this must equal $3 \sin(2x)$ we get $A = 0$ and $B = -\frac{3}{4}$.

So

$$Y_p(x) = -\frac{3}{4} x \cos(2x).$$

Hence

$$Y(x) = -\frac{3}{4} x \cos(2x) + C \sin(2x) + D \cos(2x).$$

$$Y(0) = 0 \Rightarrow D = 0.$$  

$$Y'(0) = 0 \Rightarrow -\frac{3}{4} + 2C = 0 \Rightarrow C = \frac{3}{8}.$$  

So

$$Y(x) = -\frac{3}{4} x \cos(2x) + \frac{3}{8} \sin(2x).$$
\[
\begin{aligned}
\begin{cases}
\dot{x} - 4\dot{x} + 3x &= t^2 + 1 \\
x(0) &= 1 \\
\dot{x}(0) &= 3
\end{cases}
\end{aligned}
\]

Characteristic equation: 
\[
r^2 - 4r + 3 = 0
\]
\[
(r-3)(r-1) = 0
\]
So \( r = 1 \) or \( r = 3 \)

So the solution of the homogeneous problem is
\[
x_h(t) = e^t + e^{3t}
\]

Our guess for a particular solution of the nonhomogeneous problem is
\[
x_p(t) = At^2 + Bt + C
\]
\[
\Rightarrow \dot{x}_p = 2At + B
\]
\[
\Rightarrow \ddot{x}_p = 2A
\]
\[
\Rightarrow \dddot{x}_p - 4\dot{x}_p + 3x = (2A) - 4(2At+B) + 3(At^2+Bt+C)
\]
\[
= 2A - 8At - 4B + 3At^2 + 3Bt + C
\]

This must equal \( t^2 + 1 \), so we get
\[
\begin{cases}
3A = 1 \\
3B - 8A = 0 \\
C - 4B + 2A = 1
\end{cases}
\]
The first equation gives us \( A = \frac{1}{3} \).

Plug this into the second equation to obtain
\[ B = \frac{8}{9} \text{ plug both of these into the third equation to obtain } C = \frac{35}{9}. \]

Thus the particular solution we need is

\[ X_p(t) = \frac{1}{3} t^2 + \frac{8}{9} t + \frac{35}{9}. \]

Hence

\[ X(t) = \frac{1}{3} t^2 + \frac{8}{9} t + \frac{35}{9} + C_1 e^t + C_2 e^{3t}. \]

\[ X(0) = 1 \Rightarrow \frac{35}{9} + C_1 + C_2 = 1 \]
\[ X(0) = 3 \Rightarrow \frac{8}{9} + C_1 + 3C_2 = 3 \]

So we need to solve this system of equations to obtain \[ C_1 \text{ and } C_2. \]
\[ C_1 + C_2 = \frac{-26}{9} \]
\[ C_1 + 3C_2 = \frac{19}{9} \]

The solution of this system is \[ C_1 = -\frac{53}{9}, \quad C_2 = \frac{8}{3}. \]

Therefore

\[ X(t) = \frac{1}{3} t^2 + \frac{8}{9} t + \frac{35}{9} - \frac{53}{9} e^t + \frac{8}{3} e^{3t}. \]
Suppose $A r^2 + Br + C = 0$. Let $y = e^{rt}$.

Then $y' = re^{rt}$ and $y'' = r^2 e^{rt}$, so that

$$AY'' + By' + Cy = A(r^2 e^{rt}) + B(re^{rt}) + C(e^{rt})$$

$$= e^{rt} [Ar^2 + Br + C]$$

$$= e^{rt} \cdot 0$$

$$= 0.$$ 

Let $y_1 = e^{2t}$, and suppose $y_2 = u(t)e^{2t}$ solves

$$Ay_2'' + By_2' + Cy_2 = 0.$$ 

Then

$$y_2' = u'(t)e^{2t} + 3u(t)e^{2t}$$

$$y_2'' = u''(t)e^{2t} + 6u'(t)e^{2t} + 9u(t)e^{2t}$$

Therefore

$$Ay_2'' + By_2' + Cy_2 = A(u''(t)e^{2t} + 6u'(t)e^{2t} + 9u(t)e^{2t})$$

$$+ B(u'(t)e^{2t} + 3u(t)e^{2t}) + C(u(t)e^{2t})$$

$$= Au''(t)e^{2t} + (6A + B)u'(t)e^{2t} + (9A + 3B + C)u(t)e^{2t}$$

$$= 0.$$ 

Dividing
4. Suppose \( Ar^2 + Br + c = 0 \). Let \( y = e^{rt} \).

Then \( y' = re^{rt} \) and \( y' = r^2 e^{rt} \), so that

\[
Ay'' + By' + Cy = A(re^{rt}) + B(re^{rt}) + C(e^{rt})
\]

\[
= e^{rt} \left[ Ar^2 + Br + c \right]
\]

\[
= e^{rt} \cdot 0
\]

\[
= 0. \quad \blacksquare
\]

5. Let \( y_1 = e^{3t} \) and \( y_2(t) = u(t)e^{3t} \). Then

\[
y''_2 - 6y'_2 + 9y_2 = (u''(t)e^{2t} + 6u'(t)e^{3t} + 9u(t)e^{3t})
\]

\[
- 6(u'(t)e^{2t} + 3u(t)e^{3t})
\]

\[
+ 9(u(t)e^{3t})
\]

So we want to solve \( u''(t)e^{2t} = 0 \). Divide by \( e^{3t} \)

to get \( u''(t) = 0 \). This implies \( u(t) = At + D \).

So \( y_2(t) = (At + D)e^{3t} \). Any function of this form is a solution of \( y'' - 6y' + 9y = 0 \). \( \blacksquare \)
(a) \[ 2\dddot{x} + 5\dot{x} + 4x = 19.6 \]

Homogeneous solution:
- Characteristic equation: \[ 2r^2 + 5r + 4 = 0 \]
- \[ r = \frac{-5 \pm \sqrt{25 - 32}}{4} \]
  \[ = \frac{-5 \pm \sqrt{-7}}{4} \]
- \[ x_h(t) = A e^{\frac{-5}{4}t} \sin \left( \frac{\sqrt{7}}{4} t \right) + B e^{\frac{-5}{4}t} \cos \left( \frac{\sqrt{7}}{4} t \right) \]

Particular nonhomogeneous solution:
- \[ x_p(t) = C \]
  \[ 2\dddot{x}_p + 5\dot{x}_p + 4x_p = 19.6 \]
  \[ 4C = 19.6 \]
  \[ C = 4.9 \]
- \[ x_p(t) = 4.9 \]

So,
\[ x(t) = 4.9 + A e^{\frac{-5}{4}t} \sin \left( \frac{\sqrt{7}}{4} t \right) + B e^{\frac{-5}{4}t} \cos \left( \frac{\sqrt{7}}{4} t \right) \]

Since \( e^{\frac{-5}{4}t} \to 0 \) as \( t \to \infty \), we see that
\[ \lim_{t \to \infty} x(t) = 4.9 \]
regardless of the initial conditions.

(b) \[ 2\dddot{x} + 6\dot{x} + 4x = 19.6 \]

In this situation, \[ r = \frac{-6 \pm \sqrt{36 - 32}}{4} = \frac{-6 \pm 2}{4} \], so \( r = -2 \) or \( r = -1 \).

The particular solution will be the same, so,
\[ x(t) = 4.9 + Ae^{-2t} + Be^{-t} \]
And as above,
\[ \lim_{t \to \infty} x(t) = 4.9 \]

(c) In both cases, the mass settles toward the equilibrium position that corresponds to the distance the mass would stretch the springs from its natural length.

The difference is, with a lower damping coefficient, the solution exhibits oscillation, whereas with the higher damping coefficient, there is no oscillation.
(a) \( y'' + \lambda y = 0 \), \( y(0) = 0 \), \( y(2) = 0 \)

we must have \( \lambda > 0 \) (because for \( \lambda < 0 \), there are only trivial solutions to the boundary value problem). So the general solution is

\[ y(t) = A \sin(\sqrt{\lambda} t) + B \cos(\sqrt{\lambda} t) \]

\( y(0) = 0 \) \( \Rightarrow \) \( B = 0 \)
\( \Rightarrow y(t) = A \sin(\sqrt{\lambda} t) \)

\( y(2) = 0 \) \( \Rightarrow \) \( A \sin(2\sqrt{\lambda}) = 0 \)
\( \Rightarrow \sin(2\sqrt{\lambda}) = 0 \quad \text{\( k \) is any integer} \)
\( \Rightarrow 2\sqrt{\lambda} = k\pi \)
\( \Rightarrow \sqrt{\lambda} = \frac{k\pi}{2} \)
\( \Rightarrow \lambda = \frac{k^2 \pi^2}{4} \)

\( \Rightarrow \lambda = \frac{k^2 \pi^2}{4} \quad \text{Eigenvalues} \)

For such \( \lambda \), the corresponding eigenfunctions are

\[ y(t) = A \sin\left(\frac{k\pi t}{2}\right) \]
\( b \) \( y'' + \lambda y = 0 \quad y(0) = 0 \quad y'(3) = 0 \)

Again, we need to have \( \lambda > 0 \), so

\[
\begin{align*}
\gamma(t) &= A \sin(\sqrt{\lambda} t) + B \cos(\sqrt{\lambda} t) \\
\gamma'(t) &= \sqrt{\lambda} A \cos(\sqrt{\lambda} t) - \sqrt{\lambda} B \sin(\sqrt{\lambda} t)
\end{align*}
\]

\( \gamma'(0) = 0 \quad \Rightarrow \quad A = 0 \)

\( \Rightarrow \quad \gamma'(t) = -\sqrt{\lambda} B \sin(\sqrt{\lambda} t) \)

\( \gamma'(3) = 0 \quad \Rightarrow \quad -\sqrt{\lambda} B \sin(3\sqrt{\lambda}) = 0 \)

\( \Rightarrow \quad \sin(3\sqrt{\lambda}) = 0 \)

\( \Rightarrow \quad 3\sqrt{\lambda} = k\pi \quad \text{for any integer} \)

\( \Rightarrow \quad \sqrt{\lambda} = \frac{k\pi}{3} \)

\( \Rightarrow \quad \lambda = \frac{k^2 \pi^2}{9} \quad \text{Eigenvalues} \)

For such \( \lambda \), the corresponding eigenfunctions are

\[ \gamma(t) = B \cos\left(\frac{k\pi t}{3}\right) \]