\[ \begin{cases} \frac{dy}{dx} = xy^3 \\ y(1) = -3 \end{cases} \]

Use separation of variables:

\[ \int \frac{dy}{y^3} = \int x \, dx \]

so \[ \frac{-1}{2y^2} = \frac{x^2}{2} + C \]

Hence \[ y^2 = \frac{-1}{x^2 + C} \]

so \[ y = \pm \sqrt{\frac{1}{-x^2 + C}} \]

Since \( y(1) = -3 \), we need the negative sign:

\[ y = -\sqrt{\frac{1}{-x^2 + C}} \]

and we can solve for \( C \):

\[ -3 = -\sqrt{\frac{1}{-1 + C}} \]

\[ \Rightarrow 9 = \frac{1}{-1 + C} \]

\[ \Rightarrow \frac{1}{9} = -1 + C \]

\[ \Rightarrow \frac{10}{9} = C \]

so \[ y = -\sqrt{\frac{1}{-x^2 + \frac{10}{9}}} \]

Note: we could also have solved this as a Bernoulli equation.
\[ \begin{align*}
\frac{dy}{dx} &= e^x \cos^2(y) \\
y(0) &= 0
\end{align*} \]

Use separation of variables:

\[ \int \sec^2(y) \, dy = \int e^x \, dx \]

so

\[ \tan(y) = e^x + C \]

Since \( y(0) = 0 \), we get

\[ \tan(0) = e^0 + C \]

so

\[ C = 1 + C \]

Hence \( C = -1 \). Therefore

\[ \tan(y) = e^x - 1 \]

and consequently

\[ y = \tan^{-1}(e^x - 1) + k\pi \]

Since \( y(0) = 0 \), we get \( k = 0 \). So

\[ y = \tan^{-1}(e^x - 1) \]

Therefore

\[ \lim_{x \to \infty} \tan^{-1}(e^x - 1) = \frac{\pi}{2} \]
\[ y' = y^2 + y = y(y+1) \]

Equilibrium solutions are \( y = 0 \) and \( y = -1 \).

\( y = -1 \) is stable.
\( y = 1 \) is unstable.

If \( y_0 < 0 \), then \( \lim_{t \to \infty} y(t) = -1 \).

If \( y_0 = 0 \), then \( \lim_{t \to \infty} y(t) = 0 \).

If \( y_0 > 0 \), then \( \lim_{t \to \infty} y(t) = \infty \) (assuming these limits exist).
Let $E(t)$ be the volume of ethanol in the tank after $t$ minutes. Then

\[ E(0) = 0 \quad \text{and} \quad \frac{dE}{dt} = 3 - \frac{3}{100} \cdot \frac{E}{t}. \]

We will solve this using an integrating factor:

\[ \frac{dE}{dt} + \frac{3}{100} E = 3 \]

the integrating factor is $e^{\int \frac{3}{100} dt} = e^{\frac{3t}{100}}$, so

\[ e^{\frac{3t}{100}} \frac{dE}{dt} + e^{\frac{3t}{100}} \frac{3}{100} E = 3 e^{\frac{3t}{100}} \]

or

\[ \frac{d}{dt} \left[ e^{\frac{3t}{100}} E \right] = 3 e^{\frac{3t}{100}} \]

Integrate both sides with respect to $t$:

\[ e^{\frac{3t}{100}} E = 100 e^{\frac{3t}{100}} + C \]

Using $E(0) = 0$ gives us

\[ 0 = 100 + C \]

so $C = -100$. Therefore

\[ e^{\frac{3t}{100}} E = 100 e^{\frac{3t}{100}} - 100, \]

and consequently

\[ E = 100 - 100 e^{\frac{3t}{100}} \]

After 30 minutes, the volume will be $E(30) \approx 59$ gallons. Therefore, the percentage of liquid in the tank that is ethanol will be 59%.
\[ \frac{dv}{dt} = -9.8 + kv \]

This is autonomous so we can analyze it with a phase portrait:

\begin{align*}
\text{The limit is } & \lim_{t \to \infty} v(t) = \frac{9.8}{K} \\
\text{since we're told this limit should equal } & -100, \\
\text{we get } & \frac{9.8}{K} = -100, \\
\text{so } & K = -0.098
\end{align*}

Note that we could also have answered this by solving the differential equation instead of by using the phase portrait.
$$ \frac{dp}{dt} = kP(4-P) \quad p \text{ is in millions} $$

$$ P'(0) = 0.03 \quad \text{and} \quad P(0) = 1 $$

So \quad 0.03 = k(1)(4-1) = 3k

Thus \quad k = 0.01.

Therefore we need to solve the initial value problem

$$ \begin{cases} \frac{dp}{dt} = 0.01P(4-P) \\ P(0) = 1 \end{cases} $$

Let's solve this using a substitution. Write

$$ \frac{dp}{dt} - 0.04p = -0.01p^2 $$

(note: we could have separated variables instead.)

Set \quad u = \frac{1}{p} \quad \text{so}

$$ \frac{du}{dt} = -\frac{1}{p^2} \frac{dp}{dt} = -\frac{1}{p^2} \left[ 0.04p - 0.01p^2 \right] $$

$$ = -0.04 \frac{1}{p} + 0.01 = -0.04u + 0.01 $$

Hence

$$ \frac{du}{dt} + 0.64u = 0.01 $$
Now use the integrating factor $e^{0.04t}$:

$$\frac{d}{dt} \left[ e^{0.04t} u \right] = 0.01e^{0.04t}$$

Hence

$$e^{0.04t} u = \frac{1}{4} e^{0.04t} + C$$

So

$$u = \frac{1}{4} + Ce^{-0.04t}$$

Therefore

$$P = \frac{1}{\frac{1}{4} + Ce^{-0.04t}}$$

Use $P(0) = 1$ to get

$$1 = \frac{1}{\frac{1}{4} + C}$$

$$\Rightarrow \quad 1 = \frac{1}{4} + C$$

$$\Rightarrow \quad C = \frac{3}{4}$$

So

$$P(t) = \frac{1}{\frac{1}{4} + \frac{3}{4}e^{-0.04t}}$$

We want to know when $P(t) = (0.90)(4) = 3.6$, so we solve

$$3.6 = \frac{1}{\frac{1}{4} + \frac{3}{4}e^{-0.04t}}$$

From which we get $t \approx 82$ years.
\[ \begin{align*}
\frac{dy}{dx} &= (x+y)^2 \\
y(0) &= 1
\end{align*} \]

Let's use the substitution \( u = x + y \), so that

\[ \frac{du}{dx} = 1 + \frac{dy}{dx} = 1 + (x+y)^2 = 1 + u^2. \]

Now let's separate variables:

\[ \int \frac{du}{1+u^2} = \int dx \]

So

\[ \tan^{-1}(u) = x + C. \]

Since \( y(0) = 1 \), \( u(0) = 0 + 1 = 1 \), and therefore

\[ \tan^{-1}(1) = 0 + C \]

So

\[ C = \tan^{-1}(1) = \frac{\pi}{4}. \]

Consequently

\[ \tan^{-1}(u) = x + \frac{\pi}{4} \]

So

\[ u = \tan \left( x + \frac{\pi}{4} \right) \]
9. Suppose \( A t^2 + Br + c = 0 \). Let \( y = e^{rt} \).

Then \( y' = re^{rt} \) and \( y'' = r^2 e^{rt} \), so that

\[
Ay'' + By' + Cy = A(r^2e^{rt}) + B(re^{rt}) + C(e^{rt})
\]

\[
= e^{rt} \left[ A r^2 + Br + c \right]
\]

\[
= e^{rt} \cdot 0
\]

\[
= 0. \quad \blacksquare
\]

10. Let \( y_1 = e^{3t} \) and \( y_2(t) = u(t)e^{3t} \). Then

\[
y_2'' - 6y_2' + 9y_2 = (u''(t)e^{3t} + 6u'(t)e^{3t} + 9u(t)e^{3t})
\]

\[
-6(u'(t)e^{3t} + 3u(t)e^{3t})
\]

\[
+9(u(t)e^{3t})
\]

so we want to solve \( u''(t)e^{3t} = 0 \). Divide by \( e^{3t} \) to get \( u''(t) = 0 \). This implies \( u(t) = At + D \).

So

\[
y_2(t) = (At + D)e^{3t}
\]

Any function of this form is a solution of \( y'' - 6y' + 9y = 0 \). \( \blacksquare \)