Sample Questions for Exam 2

The following list of questions is designed to give you an idea of the difficulty level of questions that I will ask on the second midterm exam. This list is not comprehensive – there are questions I could ask that are not on here. You are responsible for all the material we have covered in this course, in class, in written homework and in online quizzes. But this should serve as a guide to the level of mastery I will be looking for. This list of sample questions is slightly longer than the actual test will be.

Exam 2 will cover second order equations, including using the characteristic equation to solve homogeneous equations, using the method of undetermined coefficients to solve non-homogeneous equations, and using Laplace Transform methods. It will also cover spring-mass applications.

You will be allowed to use a single sheet (8”x11”) of notes (both sides) and a graphing calculator during the exam. No other references will be allowed.

I will not answer further questions about what will or will not be on the exam.

1. Find the general solutions of each of the following differential equations:
   (a) \(4\ddot{y} + y = 0\)
   (b) \(2\ddot{x} - 5\dot{x} = 3x\)
   (c) \(w'' + 6w' + 9w = 0\)
   (d) \(\ddot{V} - 4\dot{V} + 13V = 0\)

2. Solve the following initial value problems using the method of undetermined coefficients:
   (a) \(y'' + 4y = 3\sin(x), \ y(0) = 0, \ y'(0) = 0\)
   (b) \(y'' + 4y = 3\sin(2x), \ y(0) = 0, \ y'(0) = 0\)
   (c) \(\ddot{x} - 4\dot{x} + 3x = t^2 + 1, \ x(0) = 1, \ \dot{x}(0) = 3\)

3. The differential equation \(m\ddot{x} + \beta \dot{x} + kx = f(t)\) models the motion of a damped mass-spring system. Here, \(m\) is the mass, \(\beta\) is a damping coefficient, \(k\) is the spring constant, and \(f(t)\) is the external driving force. The function \(x(t)\) gives the displacement of the spring from its equilibrium position. Assume positive values of \(x\) represent a stretched spring, and negatives values of \(x\) represent a compressed spring.

   (a) If the only external force is gravity, pulling down on the mass to stretch the spring, then \(f(t) = mg\), where \(g\) is the acceleration due to gravity. Model such a system that has the following properties: the mass is 2 kg, the spring constant is \(4N/m\), and the damping coefficient is \(\beta = 5\). Calculate \(x(t)\) and determine \(\lim_{t\to\infty} x(t)\).
(b) Redo part (a) with the damping coefficient $\beta = 6$. Show that the limit is the same.

(c) Graph the solutions you found in (a) and (b) Describe the difference in the behavior of the mass between parts (a) and (b) above. (The situation in (a) is called underdamping, and the situation in (b) is called overdamping. Your description should shed light on these names.)

4 Use the definition of the Laplace Transform to verify each of the following rules:

(a) $\mathcal{L}[t^2] = \frac{2}{s^3}$.
(b) $\mathcal{L}[e^{at}] = \frac{1}{s-a}$
(c) $\mathcal{L}[\mathcal{U}(t-a)f(t-a)] = e^{-as}\mathcal{L}[f(t)]$

5 Let $f(t)$ be defined by:

$$ f(t) = \begin{cases} 
0 & \text{for } 0 \leq t \leq 3 \\
2 & \text{for } 3 \leq t \leq 6 \\
0 & \text{for } t \geq 6 
\end{cases} $$

Solve the initial-value problem:

$$ \begin{aligned}
    y'' - y &= f(t) \\
y(0) &= 1 \\
y'(0) &= 0
\end{aligned} $$

Write your final answer as a piecewise-defined function.

6 Consider the function:

$$ f(t) = \begin{cases} 
2 & \text{for } 0 \leq t \leq 3 \\
t & \text{for } t \geq 2 
\end{cases} $$

(a) Write $f(t)$ using step functions instead of piecewise notation.

(b) Find the Laplace Transform of $f(t)$.

The following Laplace Transform formulas will be provided for you on the exam:

$$ \mathcal{L}[1] = \frac{1}{s} $$
$$ \mathcal{L}[t^n] = \frac{n!}{s^{n+1}} $$
$$ \mathcal{L}[e^{at}] = \frac{1}{s-a} $$
$$ \mathcal{L}[\sin(kt)] = \frac{k}{s^2 + k^2} $$
$$ \mathcal{L}[^{\cos}(kt)] = \frac{s}{s^2 + k^2} $$

$$ \mathcal{L}[f'(t)] = s\mathcal{L}[f(t)] - f(0) $$
$$ \mathcal{L}[f(t-a)\mathcal{U}(t-a)] = e^{-as}\mathcal{L}[f(t)] $$

Unless a question asks you to use the definition of the Laplace Transform, you may use any of the formulas from this table without first proving them.