Generalizations of a Result of Lewis and Vogel

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Abstract

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We discuss two generalizations of the fact that a bounded domain with a well-behaved harmonic measure and a constant Poisson kernel is a ball. One generalization studies the case when the domain is unbounded and the Poisson kernel is close to a constant in a pointwise sense. The second generalization studies the bounded situation when the Poisson kernel is close to constant in the sense that it has small BMO-seminorm. *A prioiri* regularity assumptions are Ahlfors regularity and nontangential acessibility.

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LIST OF SYMBOLS

$\mathbb{R}$	the real numbers
$\mathbb{R}^n$	n-dimensional euclidean space
$\mathbf{\Omega}$	an open set in $\mathbb{R}^n$
$B(x,r)$ or $B_r(x)$	the open ball of radius r centered at x in $\mathbb{R}^n$
$\overline{B}(x,r)$ or $\overline{B}_r(x)$	the closed ball of radius r centered at x in $\mathbb{R}^n$
∂S	the topological boundary of the set $S \subset \mathbb{R}^n$
$\mathcal{L}^n$	n-dimensional Lebesgue measure
$\mathcal{H}^k$	k-dimensional Hausdorff measure
$\alpha(n)$	the Lebesgue volume of the unit ball in $\mathbb{R}^n$
σ_n	the $\mathcal{H}^n$ (surface) measure of the unit sphere in $\mathbb{R}^{n+1}$
$⨍_X f \, dx$	the average value of f over the set X
Df or ∇f	the gradient of the function f
	(this may be a classical or weak derivative, depending on context)
$D^2 f$ of $\nabla^2 f$	the hessian matrix of the function f
	(classical or weak second derivative)
$C^k(\mathbf{\Omega})$	the space of k-times continuously differentiable functions on $\mathbf{\Omega}$ for $k \geq 1$
$C(\mathbf{\Omega})$	the space of continuous functions on $\mathbf{\Omega}$
$C_c^k(\mathbf{\Omega})$	the space of functions in $C^k(\mathbf{\Omega})$ with compact support
$L^p(\mathbf{\Omega})$	the space of functions f on $\mathbf{\Omega}$ such that $\int_{\mathbf{\Omega}} \lvert f\rvert^p \, dx < \infty$
$L^p_{loc}(\mathbf{\Omega})$	the space of functions f on $\mathbf{\Omega}$ such that $\int_K \lvert f\rvert^p \, dx < \infty$
	for each compact set $K \subset \mathbf{\Omega}$

$\langle \alpha, \beta \rangle$ or $\alpha \cdot \beta$	the inner product $\sum_{i=1}^{n} \alpha_i \beta_i$ on $\mathbb{R}^n$
$diam(E)$	the diameter of the set E, given by $\sup_{x,y \in E} \lvert x - y \rvert$
spt(f)	the support of the function f

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Chapter 1

INTRODUCTION

1.1 Free Boundary Problems

A typical problem in the beginning study of partial differential examples is a boundary-value problem, and the Dirichlet problem for Laplace's equation provides one of the simplest examples: Does there exist a function u satisfying

$$\begin{cases} \Delta u = 0 & \text{in } D \\ u = g & \text{on } \partial D \end{cases} . \tag{1.1}$$

If so, what are it's regularity properties? Here, the domain D and the boundary data g are given.

A free-boundary problem turns this question inside-out. Assume that the domain D is *unknown*, but that a solution u of (1.1) is known to exist, and that it has certain regularity properties. What then can be said about the regularity or the geometry of the domain D?

The 'known properties' of u can come in a variety of forms. A typical setting occurs when u arises from a minimization problem in the calculus of variations as in the following example. Given a domain $\Omega \subset \mathbb{R}^n$, let J be the functional

$$J(u) = \int_{\Omega \cap \{u>0\}} |\nabla u|^2 + Q(x)^2 \, dx, \tag{1.2}$$

where Q is a given function and we have the side condition $u \geq 0$ on D. Here the domain of integration $\Omega \cap \{u > 0\}$ is variable. Taking the first variation for this functional tells us that a minimizer will satisfy

$$\Delta u = 0 \text{ in } \Omega \cap \{u > 0\}, \qquad u = 0 \text{ and } |\nabla u| = Q \text{ on } \Omega \cap \partial\{u > 0\}.$$

Problems like this in the calculus of variations arise in many physical situations. For example, in [1] the authors study the problem of optimizing heat flow in a steady state through a surface $\partial\Omega$ with a given finite volume of insulation on it's interior. This question may be cast in terms similar to (1.2).

The existence and regularity properties of u, and of the boundary $\partial\{u > 0\}$, for (1.2) were studied by Alt and Caffarelli in [2]. Although our point of view will be very different from this calculus-of-variations one, we mention this example in particular because the paper [2] is the source of an important technique, called non-homogeneous blow-up, of which we will make use later on. This technique also played a major role in some of the articles on which this dissertation is based.

Our approach will be to study the geometry and regularity of a domain as determined by certain properties of the harmonic measure on its boundary.

The harmonic measure for the ball $B_r(P)$ with pole at P is a constant multiple of the surface measure (n-dimensional Hausdorff measure) on $\partial B_r(P)$. In [15], Lewis and Vogel proved the converse result (with an additional hypothesis): If ω^P is harmonic measure for a bounded domain Ω with pole $P \in \Omega$ satisfying $\omega^P = c\mathcal{H}^n \llcorner \partial\Omega$, and such that

$$\omega^P(B_\rho(Q)) \leq L\rho^n \qquad \text{for all } Q \in \partial\Omega \text{ and } \rho > 0,$$

then Ω is a ball centered at P.

The additional hypothesis is needed to get a crude lower bound on the radius of a ball contained in Ω and to make careful estimates of certain integrals that describe the rate of decay of positive harmonic functions at the boundary $\partial\Omega$.

There are several directions in which one could try to generalize this result:

- What if Ω is unbounded?
- What if the harmonic measure is given by $h\mathcal{H}^n \llcorner \partial\Omega$, where the function h is *close* to a constant.

The second question here also requires that we specify what we mean by a function being 'close to a constant', and there are multiple ways one might define this.

Kenig and Toro [13] answered the first question. Preiss and Toro [16] answered the second question with the assumption that h was close to a constant in a point-wise sense. In this dissertation, we will combine these ideas to address what happens when h is close to constant in a point-wise sense *and* Ω is unbounded. We will also consider the bounded case with the point-wise assumption replaced by one that says $\log h$ has small *mean* oscillation.

1.2 History

We begin with a tour of the earlier results upon which this work is based. Readers unfamiliar with the terminology may wish to look through Section 1.3 first.

In 2001, Lewis and Vogel published in [15] that, if $\Omega \subset \mathbb{R}^{n+1}$ is a bounded domain, regular for the Dirichlet problem, containing the origin, such that the harmonic measure with pole at 0 satisfies

$$\omega^0(B_r(X)) \cap \partial\Omega) \leq Lr^n \text{ for all } X \in \partial\Omega \text{ and } 0 \leq r \leq r_0$$

and

$$\omega^0 = a\mathcal{H}^n \text{ on } \partial\Omega$$

for a positive constant a, then Ω is a ball with center at 0. It is then immediate that the radius of the ball is also determined by the constant a, since we must have

$$\int_{\partial\Omega} a\mathcal{H}^n = \omega^0(\partial\Omega) = 1,$$

so that

$$\mathcal{H}^n(\partial\Omega) = \frac{1}{a}.$$

If we now write $\Omega = B_r(0)$, we get

$$\sigma_n r^n = \frac{1}{a},$$

where $\sigma_n = \mathcal{H}^n(B_1(0))$. Therefore

$$r = (\sigma_n a)^{-\frac{1}{n}}.$$

This result was a generalization of an earlier proof (from 1992) by the same authors that had required an additional hypothesis regarding regularity of the boundary of Ω. The following is a rough sketch of the 2001 result.

The first step is to obtain a crude estimate of $|\nabla v|$ near $\partial\Omega$, where v is the Green's function for Ω with pole at 0. (When we say 'crude' in this dissertation, we mean that it is not the best estimate we will obtain – just a starting point that will lead to more refined results later.) This is done using the Riesz Representation Formula for Subharmonic Functions. The idea is that, once you have an estimate of the form $|\nabla v| \leq N$ near Ω, with $N = N(L)$, the comparison principle for harmonic functions allows you to obtain a lower bound on the radius of the largest ball centered at the origin that is contained in Ω. Let $R = \sup\{r > 0; B_r(0) \subset \Omega\}$, and let G_R denote the Green's function for $B_R(0)$ with pole at 0. Then if $Q \in \partial\Omega \cap \partial B_R(0)$, the comparison principle gives

$$|\nabla G_R(0)| \leq |\nabla v(Q)| \leq N(L);$$

but we also know

$$|\nabla G_R(0)| = \frac{1}{\sigma_n R^n},$$

so we can conclude $R \geq (\sigma_n N(L))^{-\frac{1}{n}}$. That is to say, R is bounded below in terms of L.

The next step is to define $M = \limsup_{X \to \partial\Omega} |\nabla v|$ and to prove, via contradiction, that $M \leq a$. Achieving this allows one to repeat the calculations above to get $R \geq (\sigma_n a)^{-\frac{1}{n}}$; the isoperimetric inequality then implies $\Omega = B_R(0)$. Having a crude upper bound on $|\nabla v|$ near $\partial\Omega$ is an important element of the indirect proof.

Preiss and Toro [16] generalize this result as follows: Suppose that $\Omega \subset \mathbb{R}^{n+1}$ is a bounded domain containing the origin that satisfies

$$\sup_{0<r<1} \sup_{Q \in \partial\Omega} \frac{\mathcal{H}^n(B_r(Q) \cap \partial\Omega)}{r^n} < \infty.$$

Then given $\epsilon > 0$ small enough, if the Poisson kernel h for Ω with pole at 0 exists and satisfies

$$\sup_{\partial\Omega} |\log h| < \epsilon,$$

then

$$B_{R_1}(0) \subset \Omega \subset B_{R_2}(0)$$

with

$$e^{-\epsilon} \leq \sigma_n R_1^n \leq \sigma_n R_2^n \leq e^{\epsilon}.$$

One may think of this as a stability result for the theorem of Lewis and Vogel: a small perturbation of the Poisson kernel from constant results in only a small geometric perturbation of Ω from a ball.

The paper [16] actually goes further than this geometric result. The authors also prove a regularity result for the boundary $\partial\Omega$: if $\delta > 0$ is sufficiently small

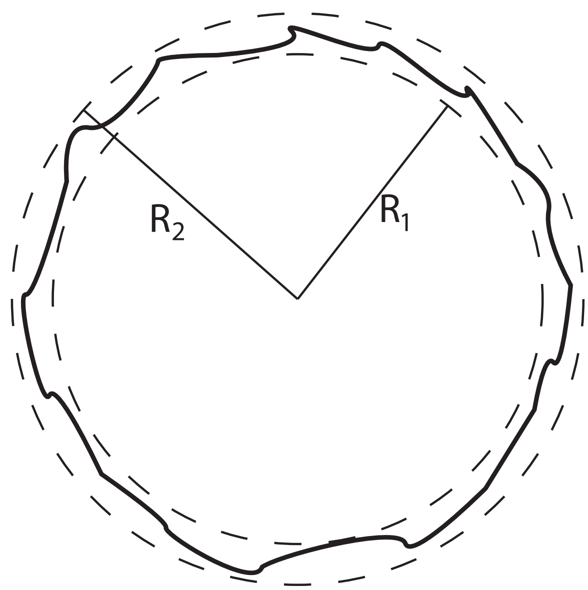

Figure 1.1: The boundary $\partial\Omega$ sits between the two spheres with radii R_1 and R_2.

there exists $\epsilon > 0$ so that, under the same conditions as above, Ω is δ-Reifenberg flat. (See Definition (9) below.)

That argument is based in large part on the techniques in [2]. The idea there is to define *flatness* of the boundary at a point $Q \in \partial\Omega$ in terms of the linear-growth behavior of the Green's function and the associated Poisson kernel near Q. Having done this, one can use the theory of partial differential equation to improve estimates, and thereby improve the measure of *flatness* at Q in successively smaller neighborhoods. Alt and Caffarelli employ this argument to show that, if the function Q used in the functional (1.2) is Holder continuous, then the free-boundary $\partial\{u > 0\} \cap \Omega$ of the minimizer u is in fact smooth.

That technique is modified slightly in [16]. The setup there does not allow the estimates to improve dramatically as one looks at successively smaller neighborhoods of Q. But the estimates do *persist* as one looks at successively smaller neighborhoods of Q, so the measure of flatness at a larger scale can be duplicated at all smaller scales.

Heuristically, the idea in [2] is that “flatness at one scale implies greater flatness at a smaller scale”; in [16] the idea is that “flatness at one scale implies similar flatness at all smaller scales.”

Kenig and Toro [13] develop a generalization of the results in [2] and, in the process, obtain a generalization of Lewis and Vogel along different lines. Instead of considering a bounded domain, the authors asked what happens when Ω is unbounded. They proved that there exists $\delta_n > 0$ such that if $\Omega \subset \mathbb{R}^{n+1}$ is an unbounded δ-Reifenberg flat chord arc domain (for $\delta \in (0, \delta_n)$), the Green's function with pole at ∞, v, and the corresponding Poisson kernel, h, satisfy

$$\sup_{X\in\Omega} |\nabla v(X)| \leq 1 \qquad \text{and} \qquad h(Q) \geq 1 \text{ for } \mathcal{H}^n \; a.e. \; Q \in \partial\Omega,$$

then Ω is a half space, and in suitable coordinates, $v(x, x_{n+1}) = x_{n+1}$.

To summarize, given very loose assumptions about the growth properties of harmonic measure on Euclidean balls, Lewis and Vogel proved that constant Poisson kernels correspond to balls for bounded domains, while Kenig and Toro showed that they correspond to half spaces for unbounded domains.

Again, a key idea was to use the “flat at one scale implies greater flatness at smaller scales” argument described in [2].

The generalization sought in Chapter 2 of this dissertation combines these ideas. We assume that $\Omega \subset \mathbb{R}^{n+1}$ is an unbounded domain, and that it's Poisson kernel is not much less than 1, while the Green's function maintains a linear growth near the boundary; and we assume that the boundary is ‘flat’ at *very large scales*. We also require that $\partial\Omega$ be Ahlfors regular and that the harmonic measure be well-behaved: $\omega(B_r(Q)) \leq Lr^n$ for all balls $B_r(Q)$ centered on $\partial\Omega$. We use procedures similar to those in [13] to prove that the boundary is also flat at all small scales. This result is a quantitative version of [13].

Chapter 3 of this dissertation takes a very different approach to showing stability of the result of Lewis and Vogel. We start with a bounded domain. However, instead of perturbing the Poisson kernel h from constant in a *point-wise* sense, as in [16], we perturb it in an *average sense* by assuming that $\log h$ is a function of bounded mean oscillation (BMO) with small BMO-seminorm.

Our argument will begin as did Lewis and Vogel's, with a crude estimate on the gradient of a Green's function near $\partial\Omega$. However, to improve the gradient estimate, the processes in [15] and [16] make explicit use of point-wise properties of h which we will not have assumed. Instead, we modify gradient estimates used in [14] when studying Poisson kernels of bounded mean oscillation.

As above, a comparison principle argument allows us to turn a gradient estimate for the Green's function into a lower bound for the radius of the largest ball $B_R(0)$ contained in Ω.

Moving from there to an estimate on the radius of a ball containing Ω again requires a different approach than is used in any of these other papers. In [15] and [16], the authors were able to make use of their knowledge of the point-wise behavior of h to estimate the total surface measure $\mathcal{H}^n(\partial\Omega)$; then another comparison principle argument in [16] or an application of the isoperimetric inequality in [15] completes the argument. With only a hypothesis about the average behavior of h, we will not be able to use either approach.

Instead, we use a kind of 'piecewise projection' of $\partial\Omega$ onto concentric balls around the origin to make an estimate of how much of the boundary is far from the 'inner ball' already discovered. That quantity cannot be too large, and then using the assumption that $\partial\Omega$ is Ahlfors regular, we manage to conclude that no point of $\partial\Omega$ can be very far from that inner ball.

The process that goes from "flatness at a large scale" to "flatness at smaller scales" does not seem to work here, however, because it requires some knowledge of the point-wise behavior of h. Therefore the results in Chapter 3 are purely geometric (Ω is 'close to being a ball'), and we do not discuss regularity.

1.3 Preliminaries

Because we will typically be concerned with the boundary of a domain, it will be most convenient for us to consider the boundary to be n-dimensional and therefore that our domains be open subsets of $\mathbb{R}^{n+1}$.

1.3.1 Geometric Measure Theory

Definition 1. *The* **Hausdorff distance** *between two nonempty sets* $A, B \subset \mathbb{R}^{n+1}$ *is defined to be*

$$D[A, B] = \sup_{a \in A} dist(a, B) + \sup_{b \in B} dist(b, A).$$

Notice that the Hausdorff distance between two closed sets A and B is zero if and only if $A = B$. This notion of distance provides a metric on the class $\mathcal{C}_{\mathcal{R}}$ of nonempty compact subsets of $\overline{B_R(0)} \subset \mathbb{R}^{n+1}$; in fact, $(\mathcal{C}_{\mathcal{R}}, D)$ is a compact metric space. In particular, Cauchy sequences have limits: if A_k is an infinite sequence in $\mathcal{C}$ and for all $\epsilon > 0$, $D[A_l, A_m] < \epsilon$ for $l, m \geq N(\epsilon)$, then there is a compact set $A \in \mathcal{C}$ such that $D[A_k, A] \to 0$ as $k \to \infty$. The proof of this fact is a standard exercise in the theory of metric spaces.

Definition 2. *The* **k-dimensional Hausdorff measure** *of a set* E *in Euclidean space is*

$$\mathcal{H}^k(E) = \lim_{\epsilon \searrow 0} \inf \left\{ \sum_{i=1}^{\infty} c_k (diam(E_i))^s ; E \subset \bigcup_{i=1}^{\infty} E_i \textit{ and } 0 \leq diam(E_i) < \epsilon \right\},$$

where the constants c_k *are chosen so that* $\mathcal{H}^k$ *agrees with* k*-dimensional Lebesgue measure on* $\mathbb{R}^k$*:*

$$\mathcal{H}^k(B_r(0)) = r^k \int_{B_1(0)} dx \quad \textit{for } B_1(0) \subset \mathbb{R}^k.$$

(We will have no need here for Hausdorff measure of fractional dimension.) We used here the diameter of a set $E \subset \mathbb{R}^{n+1}$:

$$diam(E) = \sup_{X,Y \in E} |X - Y|.$$

The notation $\mathcal{H}^k \llcorner S$ means the measure is restricted to a set S:

$$(\mathcal{H}^k \llcorner S)(E) = \mathcal{H}^k(S \cap E).$$

Note that $\mathcal{H}^n \llcorner \partial\Omega$ is surface measure on $\partial\Omega$ for a smooth domain in $\mathbb{R}^{n+1}$.

Definition 3. *A domain* $\Omega \subset \mathbb{R}^{n+1}$ *is said to have* **finite perimeter** *if*

$$\sup \left\{ \int_\Omega div\ \phi\ dx; \phi \in C^1_c(\mathbb{R}^{n+1}; \mathbb{R}^{n+1}), |\phi| \le 1 \right\} < \infty.$$

When $\partial\Omega$ is smooth, this supremum coincides with surface measure because

$$\begin{aligned} \int_\Omega \text{div}\ \phi\ dx &= \int_{\partial\Omega} \phi \cdot \vec{\nu}\ d\mathcal{H}^n \\ &\le \int_{\partial\Omega} d\mathcal{H}^n \quad \text{(since } |\phi| \le 1) \\ &= \mathcal{H}^n(\partial\Omega), \end{aligned}$$

and we obtain equality by choosing ϕ to agree with the outward unit normal vector field $\vec{\nu}$ along $\partial\Omega$. An advantage of this definition is that it makes sense for any $\Omega \subset \mathbb{R}^{n+1}$ – no *a priori* regularity need be assumed.

Definition 4. *A set* $\Omega \subset \mathbb{R}^{n+1}$ *is said to have* **locally finite perimeter** *if for each open set* $V \subset \mathbb{R}^{n+1}$ *with compact closure, we have*

$$\sup \left\{ \int_{V \cap \Omega} div\ \phi\ dx; \phi \in C^1_c(V; \mathbb{R}^{n+1}), |\phi| \le 1 \right\} < \infty.$$

As a consequence of the Riesz Representation Theorem, if Ω has locally finite perimeter, there is a Radon measure μ on $\mathbb{R}^{n+1}$ and a μ-measurable function $\nu : \partial\Omega \to \mathbb{R}^{n+1}$ such that

$$|\nu(X)| = 1 \text{ for } \mu - a.e. X \in \partial\Omega, \text{ and}$$

$$\int_\Omega \text{div}\, \phi\, dx = -\int_{\mathbb{R}^{n+1}} \phi \cdot \nu\, d\mu \text{ for all } \phi \in C_c^1(\mathbb{R}^{n+1}; \mathbb{R}^{n+1}).$$

Note that this appears to be a generalization of the divergence theorem, except that we do not yet have much information about the measure μ or its support.

Definition 5. *If Ω is a set of locally finite perimeter, we say that $X \in \partial^*\Omega$, the* **reduced boundary** *of Ω, if*

1. $\mu(B_r(X)) > 0$ *for all* $r > 0$,
2. $\lim_{r\to 0} ⨍_{B_r(X)} \sigma d\mu = \sigma(X)$, *and*
3. $|\nu(X)| = 1$,

with ν and μ as above.

By the Lebesgue-Besicovitch Differentiation Theorem (section 1.7.1 in [5],

$$\mu(\partial\Omega - \partial^*\Omega) = 0.$$

Blow-ups of the reduced boundary lead to half spaces: If $X \in \partial^*\Omega$, define

$$\Omega_r = \{Y \in \mathbb{R}^n; r(Y - X) + X \in \Omega\}$$

and

$$H^-(X) = \{Y \in \mathbb{R}^{n+1}; \sigma(X) \cdot (Y - X) \leq 0\}.$$

Then

$$\chi_{\Omega_r} \to \chi_{H^-(X)} \text{ in } L^1_{loc}(\mathbb{R}^{n+1}) \text{ as } r \searrow 0.$$

(See Section 5.7.2 of [5].) Furthermore,

$$\nu = \mathcal{H}^n \llcorner \partial^* \Omega.$$

This implies $\mathcal{H}^n(\partial\Omega \cap K) < \infty$ for each compact set $K \subset \mathbb{R}^{n+1}$.

Definition 6. *Let* $X \in \mathbb{R}^{n+1}$. *We say that* $X \in \partial_* \Omega$, *the* **measure theoretic boundary** *of* Ω, *if*

$$\limsup_{r \searrow 0} \frac{\mathcal{L}^{n+1}(B_r(X) \cap \Omega)}{r^{n+1}} > 0$$

and

$$\limsup_{r \searrow 0} \frac{\mathcal{L}^{n+1}(B_r(X) \setminus \Omega)}{r^{n+1}} > 0.$$

Lemma 1 in section 5.8 of [5] shows that $\partial^*\Omega \subset \partial_*\Omega$ and $\mathcal{H}^n(\partial_*\Omega \setminus \partial^*\Omega) = 0$. Theorem 1 in the same section then gives the full generalization of the divergence theorem: If $\Omega \subset \mathbb{R}^{n+1}$ has locally finite perimeter, then $\mathcal{H}^n(\partial\Omega \cap K) < \infty$ for each compact $K \subset \mathbb{R}^{n+1}$; and, for $\mathcal{H}^n$ a.e. $X \in \partial_*\Omega$, there is a unique measure-theoretic unit outer normal vector $\nu_\Omega(X)$ such that

$$\int_E \operatorname{div} \phi \, dx = \int_{\partial_*\Omega} \phi \cdot \nu_\Omega \, d\mathcal{H}^n$$

for all $\phi \in C_c^1(\mathbb{R}^{n+1}; \mathbb{R}^{n+1})$.

Definition 7. *A domain* $\Omega \subset \mathbb{R}^{n+1}$ *is said to be* **Ahlfors regular** *if there is a constant* $A \geq 1$ *such that, for all* $Q \in \partial\Omega$ *and all* $r \in (0, diam(\Omega))$,

$$\frac{r^n}{A} \leq \mathcal{H}^n(\partial\Omega \cap B_r(Q)) \leq A r^n.$$

Observe that an Ahlfors regular domain has locally finite perimeter.

Definition 8. *A domain* $\Omega \subset \mathbb{R}^{n+1}$ *is said to have the* **separation property** *if for each compact set* $K \subset \mathbb{R}^{n+1}$ *there exists* $R > 0$ *such that for* $Q \in \partial\Omega \cap K$ *and*

$r \in (0, R]$ there exists an n-dimensional plane $L(Q,r)$ containing Q and a choice of unit normal vector to $L(Q,r)$, $\vec{n_{Q,r}}$, satisfying

$$\mathcal{T}^+(Q,r) = \left\{ X = x + t\vec{n_{Q,r}} \in B_r(Q); x \in L(Q,r) \text{ and } t > \frac{1}{4}r \right\} \subset \Omega$$

and

$$\mathcal{T}^-(Q,r) = \left\{ X = x + t\vec{n_{Q,r}} \in B_r(Q); x \in L(Q,r) \text{ and } t < \frac{1}{4}r \right\} \subset \Omega^c.$$

Moreover, if Ω is an unbounded domain we also require that $\partial\Omega$ divide $\mathbb{R}^{n+1}$ into two distinct connected components Ω and $\Omega^c \neq \emptyset$.

Definition 9. *Let $\delta > 0$ be small, and let $\Omega \subset \mathbb{R}^{n+1}$ be a set of locally finite perimeter. We say that Ω is a δ-**Reifenberg flat chord arc domain** or a **Reifenberg flat chord arc domain** if*

1. *Ω has the separation property.*

2. *For each compact set $K \subset \mathbb{R}^{n+1}$, there exists $R_K > 0$ such that for every $Q \in K \cap \partial\Omega$ and every $r \in (0, R_K]$,*

$$\inf_L \left\{ \frac{1}{r} D[\partial\Omega \cap B_r(Q), L \cap B_r(Q)] \right\} \leq \delta,$$

   where the infemum is taken over all n-planes through Q. Moreover if Ω is unbounded we require $R_K = \infty$.

3. *$\partial\Omega$ is Ahlfors regular.*

1.3.2 Harmonic Functions

Definition 10. *A bounded domain $\Omega \subset \mathbb{R}^{n+1}$ is said to be **regular** for the Dirichlet problem if, for every continuous function $g \in C(\partial\Omega)$, there is a solu-*

tion u of the boundary-value problem

$$\begin{cases} \Delta u = 0 & \textit{in } \Omega \\ u = g & \textit{on } \partial\Omega \\ u \in C^2(\Omega) \cap C(\overline{\Omega}) \end{cases} \tag{1.3}$$

The class of regular domains is very general (see chapter 2 of [6]). Here, Δ is the Laplace operator: $\Delta u = \frac{\partial^2 u}{\partial x_1^2} + ... + \frac{\partial^2 u}{\partial x_{n+1}^2}$; solutions of $\Delta u = 0$ are called **harmonic** functions. Notice that, if u_1 and u_2 solve (1.3) for the boundary data $g = g_1$ and $g = g_2$, respectively, then $au_1 + bu_2$ solves (1.3) for $g = ag_1 + bg_2$ (where a and b are any constants). Harmonic functions satisfy the **weak maximum principle**:

$$\sup_{X \in \Omega} u = \sup_{X \in \partial\Omega} u.$$

They also satisfy the **strong maximum principle**: if $u(P) = \sup_{\partial\Omega} u$ for some $P \in \Omega$, then u is constant on the connected component of Ω containing P. We will refer to either of these in this dissertation as a **maximum principle**. In particular, they imply that, for a connected domain Ω, solutions of (1.3) are unique. Therefore, for any $P \in \Omega$, the mapping

$$g \in C(\partial\Omega) \to u$$

is linear, and so for fixed $P \in \Omega$,

$$g \in C(\partial\Omega) \to u(P) \tag{1.4}$$

defines a linear functional on $C(\Omega)$. Furthermore, if we equip $C(\partial\Omega)$ with the uniform norm, $\| \ g \ \| = \sup_{\partial\Omega} |g|$, then we see that (1.4) is actually a bounded linear functional because the strong maximum principle gives us $|u(P)| \leq \| \ g \ \|$.

Consequently, the Riesz Representation Theorem (see Theorem 6.19 in [17]) tells us that there is a probability measure ω^P defined on $\partial\Omega$ such that

$$u(P) = \int g \, d\omega^P. \tag{1.5}$$

Definition 11. *The measure defined by (1.5) is called the* **harmonic measure for Ω with pole P.**

Definition 12. *For $n \geq 2$, the function*

$$\Phi(X) = \frac{1}{(n-1)\sigma_n |X|^{n-1}} \tag{1.6}$$

is called the **fundamental solution** *of Δ.*

The fundamental solution satisfies $\Delta\Phi = -\delta_0$ in the sense of distributions, where δ_Y is the point mass at the point Y. That is to say, for any $\eta \in C_c^\infty(\mathbb{R}^{n+1})$ we have

$$\int_{\mathbb{R}^{n+1}} \eta(Y)\Phi(Y)\, dY = -\eta(0).$$

This last equality can be verified directly by an argument using the divergence theorem and integration-by-parts. (See Chapter 2 of [4].) (When $n = 1$, the function $F(X) = \ln|X|$ provides a fundamental solution; but in this dissertation we will only be concerned with the case $n \geq 2$.)

If Ω is regular for the Dirichlet problem and $P \in \Omega$, then there is a solution u_P of the boundary-value problem

$$\begin{cases} \Delta u_P = 0 & \text{in } \Omega \\ u_P(X) = F(X-P) & \text{for } X \in \partial\Omega \\ u_P \in C^2(\Omega) \cap C(\overline{\Omega}) \end{cases}$$

Then the function G_P defined on $\overline{\Omega} \setminus \{P\}$ by $G(X) = F(X-P) - u_P(X)$ satisfies

$$G_P = 0 \text{ on } \partial\Omega, \quad G_P > 0 \text{ on } \Omega \setminus \{P\} \quad \text{and} \quad \Delta G_P = \delta_P.$$

Definition 13. *G_P is called the* **Green's function** *for Ω with pole at P.*

We typically extend G to be a continuous function on $\mathbb{R}^{n+1} \setminus \{P\}$ by setting $G = 0$ on Ω^c.

If Ω is a C^1 domain (i.e. the boundary $\partial\Omega$ is locally the graph of a continuously differentiable function), then it turns out that the normal derivative $\frac{\partial G_P}{\partial \nu}$, where ν is the *inward* unit normal vector along Ω, provides us with a means to write down the harmonic measure for Ω with pole at P; then

$$d\omega^P = \frac{\partial G_P}{\partial \nu} d\mathcal{H}^n \llcorner \partial\Omega. \tag{1.7}$$

The proof of this fact is an application of the divergence theorem.

It is important to note here that Ω need not be a C^1 domain for the Green's function and the harmonic measure to exist. But as long as Ω is a domain for which a general divergence theorem holds, then something like (1.7) will hold. (See the notion of domains of locally-finite perimeter, defined above.)

Suppose now that Ω has locally finite perimeter, so that differentiation with respect to $\mathcal{H}^n \llcorner \partial\Omega$ makes sense. If it turns out that if ω^P is absolutely continuous with respect to $\mathcal{H}^n$ on $\partial\Omega$, then the Lebesgue-Radon-Nikodym derivative $h_P = \frac{d\omega^P}{d\mathcal{H}^n}$ exists, is nonnegative, is $\mathcal{H}^n$-measurable, is unique up to a set of $\mathcal{H}^n$-measure zero, and satisfies

$$d\omega^P = h_P d\mathcal{H}^n.$$

Definition 14. *The function h_P is called the* **Poisson kernel** *for Ω with pole at P.*

It satisfies

$$u(P) = \int_{\partial\Omega} g h_P \, d\mathcal{H}^n$$

for u and g as in (1.3). Because ω^P is a probability measure and $h_P \geq 0$

$$\int_{\partial\Omega} h_P \, d\mathcal{H}^n = \int_{\partial\Omega} d\omega^P = 1,$$

so we also have $h_P \in L^1(\partial\Omega; \mathcal{H}^n)$.

EXAMPLE: Let $\Omega = B_R(0) \subset \mathbb{R}^{n+1}$. Then

$$G(X) = \frac{1}{(n-1)\sigma_n |X|^{n-1}} - \frac{1}{(n-1)\sigma_n R^{n-1}}$$

is a Green's function for Ω with pole at the origin. Harmonic measure with pole at the origin turns out to be a multiple of surface measure:

$$\omega^0(E) = \frac{\mathcal{H}^n(E)}{\mathcal{H}^n(\partial B_R(0))}.$$

(This is a result of the mean value theorem for harmonic functions: see Chapter 2 of [4].) The corresponding Poisson kernel is therefore constant:

$$h_P(Q) = \frac{1}{\mathcal{H}^n(\partial B_R(0))} \qquad \text{for all } Q \in \partial\Omega.$$

□

1.3.3 Non-tangentially Accessible Domains

The notion of a (bounded) non-tangentially accessible (NTA) domain was introduced in the 1982 article [9] by Jerison and Kenig. In that paper, the authors generalize the classical theory of boundary behavior of harmonic functions known previously for Lipschitz domains (see the 1970 article [8]). NTA domains are much more general than Lipschitz domains, but they maintain many of the classical properties of Lipschitz domains.

For example, it was shown by Calderon in 1950 (in [3]) that if u is a harmonic function in $\mathbb{R}^{n+1}_+$ which is non-tangentially bounded at every point of a measurable set $E \subset \partial\mathbb{R}^n_+$, then u has a non-tangential limit at almost every x in E. In a 1961 article [18], Stein posed the question of extending these results (and others) to the most general domains "for which non-tangential behavior is meaningful." Hunt and Wheeden [8] took up the challenge for Lipschitz domains, and Jerison and Kenig extended the results to their even more general NTA domains.

Definition 15. *A ball $B_r(X) \subset \Omega$ is called M-***non-tangential** *if*

$$\frac{M+1}{M} r \leq dist(X, \partial\Omega) \leq (M+1)r.$$

This is equivalent to saying that $M^{-1} \leq \frac{d}{r} \leq M$, where d is the distance of the ball $B_r(X)$ from the boundary $\partial\Omega$.

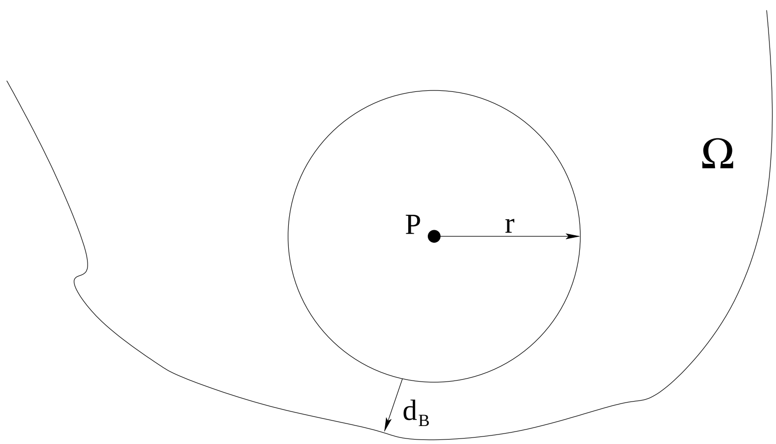

Figure 1.2: An M-non-tangential ball with center P, radius r and distance d_B from the boundary.

Suppose that $B_{r_1}(P_1)$ and $B_{r_2}(P_2)$ are M-non-tangential balls with $B_{r_1}(P_1) \cap B_{r_2}(P_2) \neq \emptyset$. Then

$$\frac{M+1}{M} r_2 \leq d(P_2, \partial\Omega) \leq |P_1 - P_2| + d(P_1, \partial\Omega) \leq r_1 + r_2 + (M+1)r_1,$$

and thus $r_2 \leq M(M+2)r_1$. Switching the roles of r_1 and r_2 yields

$$\frac{1}{\tilde{M}} r_1 \leq r_2 \leq \tilde{M} r_1 \quad \text{for} \quad \tilde{M} = M(M+2). \tag{1.8}$$

That is to say, M-non-tangential balls that meet have comparable radii, with comparison constant $\tilde{M}$.

Definition 16. *If $X_1, X_2 \in \Omega$, a* **Harnack chain** *from X_1 to X_2 is a collection of M-non-tangential balls $B_1, ..., B_k$ such that $X_1 \in B_1$, $X_2 \in B_k$ and $B_i \cap B_{i+1} \neq \emptyset$*

for all $i = 1, ..., k-1$. The **length** *of the chain, k, is the number of balls in the collection.*

Definition 17. *A bounded domain $\Omega \subset \mathbb{R}^N$ is said to be* **non-tangentially accessible** *(or* **NTA** *for short) if there exist $M > 0$ and $r_0 > 0$ such that the following conditions are satisfied:*

1. **(Interior Corkscrew Condition)** *For all $Q \in \partial\Omega$ and every $r \in (0, r_0)$ there exists a point $A_r(Q) \in \Omega$ satisfying*

$$\frac{r}{M} \leq dist(A_r(Q), \partial\Omega) \leq |A_r(Q) - Q| \leq r.$$

2. **(Exterior Corkscrew Condition)** *For all $Q \in \partial\Omega$ and every $r \in (0, r_0)$ there exists a point $\tilde{A}_r(Q) \in \Omega^c$ satisfying*

$$\frac{r}{M} \leq dist(\tilde{A}_r(Q), \partial\Omega) \leq |\tilde{A}_r(Q) - Q| \leq r.$$

3. **(Harnack Chain Condition)** *For every $X_1, X_2 \in \Omega$, if*

$$\epsilon \leq \min\{dist(X_1, \partial\Omega), dist(X_2, \partial\Omega)\}$$

and

$$|X_1 - X_2| \leq 2^k \epsilon,$$

then there is a Harnack Chain in Ω from X_1 to X_2 of length at most Mk.

Unbounded NTA domains will be defined below.

Remarks: The definition here for bounded domains is slightly stronger than the one given in [9]. In that article, the authors do not require that the length of the Harnack chain be bounded by Mk, only that the length depends on k but not ϵ. However, in the intervening years, the definition given here has become the working definition in the field. See, for example, [10]. The points

$A_r(Q)$ and $\tilde{A}_r(Q)$ are called **M-non-tangential points relative to** Q, and the constants M and r_0 are referred to as the **NTA constants of** Ω Observe that the balls $B_{\frac{r}{2M}}(A_r(Q))$ are 3M-non-tangential.

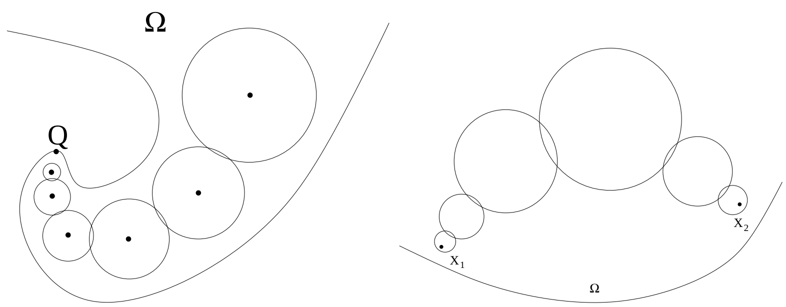

Figure 1.3: Some non-tangential balls in an interior corkscrew, and a Harnack Chain.

The Exterior Corkscrew Condition allows one to construct 'barriers' whose existence prove that these domains are regular for the Dirichlet problem. The other two conditions are essential to producing certain estimates about the harmonic measures supported on the boundaries of NTA domains. In essence, these conditions generalize the main properties of smooth and Lipschitz domains which lead to doubling properties of the harmonic measure. Indeed, the following types of bounded domains are all subsets of the class of NTA domains:

- Smooth Domains
- Lipschitz Domains
- Zygmund Domains
- Quasispheres

A quasisphere in $\mathbb{R}^{n+1}$ is the image of $B_1(0) \subset \mathbb{R}^{n+1}$ under a quasi-conformal mapping $\mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$. A Zygmund domain is a domain whose boundary is

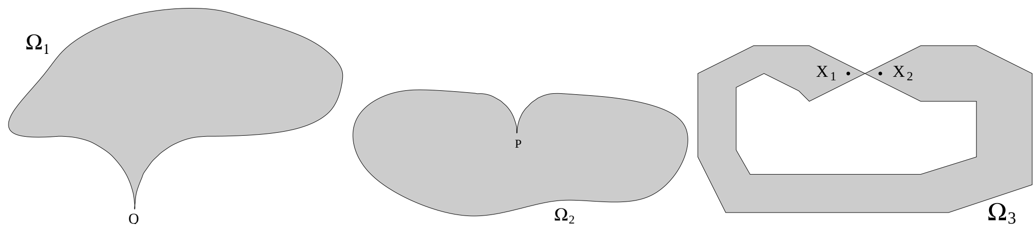

Figure 1.4: Not NTA Domains

locally the graph of a Zygmund-class function, i.e. a function in the family

$$\Lambda_1(\mathbb{R}^n) = \left\{ \phi : \mathbb{R}^n \to \mathbb{R};\ \sup_{x,z\in\mathbb{R}^n} \frac{|\phi(x+z)+\phi(x-z)-2\phi(x)|}{|z|} < \infty \right\}.$$

Obviously C^1 functions are Zygmund-class, but so are some nowhere-differentiable functions, including the Weierstrass function

$$\phi(x) = \sum_{n=1}^{\infty} \frac{\cos(3^n x)}{2^n}.$$

In order to better understand which domains are NTA, we illustrate in Figure 1.4 some domains which fail these hypotheses. The domain Ω_1 fails the Interior Corkscrew Condition at Q; Ω_2 fails the Exterior Corkscrew Condition at P; and Ω_3 fails the Harnack Chain Condition, which we see by letting X_1 and X_2 get closer to each other on opposite sides of the vertex.

Non-tangentially accessible domains were developed in [9] to generalize what was know about the behavior of harmonic functions on Lipschitz domains due to Hunt and Wheeden [8]. We collect some of those important facts here as the following three lemmas.

Lemma 1. *Harmonic measure on an NTA domain is a doubling measure: If Ω is NTA, $X \in \Omega$ and ω^X is the harmonic measure for Ω with pole at X, then*

$$\omega^X(B_{2r}(Q)) \le C_X \omega^X(B_r(Q)).$$

This is proved as Lemma 4.9 in [9]. The next fact is Lemma 4.1 in [9].

Lemma 2. *Positive harmonic functions that vanish continuously on the boundary of an NTA domain do so in a Hölder continuous fashion: If Ω is NTA with constants M and r_0, there exists $\beta > 0$ such that for all $Q \in \partial\Omega$, $r < r_0$ and every positive harmonic function u in Ω, if u vanishes continuously on $\partial\Omega \cap B_r(Q)$, then for $X \in \Omega \cap B_r(Q)$,*

$$u(X) \leq M(|X - Q|r^{-1})^{\beta} C(u),$$

where $C(u) = \sup\{u(Y); Y \in \partial B_r(Q) \cap \Omega\}$.

The main estimate we will need when working with NTA domains is a relationship between the harmonic measure of a ball and the value of the Green's function nearby (Lemma 4.8 in [9]):

Lemma 3. *There exists $C = C(M) > 1$ such that*

$$C^{-1} < \frac{\omega^Y(B_\delta(x)(Q))}{\delta(X)^{n-1} G_X(Y)} < C,$$

where ω^Y is harmonic measure with pole at Y, M is the NTA constant of Ω, G_X is the Green's function for Ω with pole at X, and $\delta(X) = \inf\{|X - P|; P \in \partial\Omega\}$.

1.3.4 Unbounded Domains

As mentioned previously, bounded NTA domains were introduced the the 1982 paper [9], but unbounded NTA domains were not introduced until 1999 in [11] by Kenig and Toro. The definition is essentially the same, except we drop the constant r_0 and require the interior and exterior corkscrew conditions to hold for all distances r from the boundary. The modification is necessary in order to ensure a desired doubling property for the harmonic measure of unbounded Ω and a Harnack inequality.

Figure 1.5: Not Unbounded NTA

Definition 18. *An unbounded domain* $\Omega \subset \mathbb{R}^N$ *is said to be* **non-tangentially accessible** *(or* **NTA** *for short) if there exist* $M > 0$ *such that the following conditions are satisfied:*

1. **(Interior Corkscrew Condition)** *For all* $Q \in \partial\Omega$ *and every* $r > 0$ *there exists a point* $A_r(Q) \in \Omega$ *satisfying*

$$\frac{r}{M} \leq dist(A_r(Q), \partial\Omega) \leq |A_r(Q) - Q| \leq r.$$

2. **(Exterior Corkscrew Condition)** *For all* $Q \in \partial\Omega$ *and every* $r > 0$ *there exists a point* $\tilde{A}_r(Q) \in \Omega^c$ *satisfying*

$$\frac{r}{M} \leq dist(\tilde{A}_r(Q), \partial\Omega) \leq |\tilde{A}_r(Q) - Q| \leq r.$$

3. **(Harnack Chain Condition)** *For every* $X_1, X_2 \in \Omega$, *if*

$$\epsilon \leq \min\{dist(X_1, \partial\Omega), dist(X_2, \partial\Omega)\}$$

and

$$|X_1 - X_2| \leq 2^k \epsilon,$$

then there is a Harnack Chain in Ω *from* X_1 *to* X_2 *of length at most* Mk.

The unbounded smooth domain Ω_4 in Figure 1.5 is not an NTA domain because it fails the interior corkscrew condition: non-tangential balls do not have enough room to grow proportionally to their distance from Q as that distance increases without bound.The complement of $\overline{\Omega_4}$ would also not be NTA because it would fail the exterior corkscrew condition.

We can also discuss Green's functions, harmonic measures and Poisson kernels for unbounded domains, except that these ideas become more complicated since solutions of the Dirichlet problem need not be unique on unbounded domains (because the maximum principle does not apply). In particular, the map (1.4) is not well-defined, so we must be more direct.

Definition 19. *Let $\Omega \subset \mathbb{R}^{n+1}$ be an unbounded domain. We say that a continuous function $G : \overline{\Omega} \to \mathbb{R}$ is a* **Green's function with pole at** ∞ *for Ω if*

$$\begin{cases} G > 0 & \textit{in } \Omega \\ G = 0 & \textit{on } \partial\Omega \\ \Delta G = 0 & \textit{in } \Omega \end{cases}$$

Definition 20. *If Ω has locally finite perimeter, an associated* **Poisson kernel** *for Ω with pole at ∞ is a function h on $\partial\Omega$ such that for all $\phi \in C_c^\infty(\mathbb{R}^{n+1})$ we have*

$$\int_{\mathbb{R}^{n+1}} G\Delta\phi \, dx = \int_{\partial\Omega} \phi h \, d\mathcal{H}^n.$$

Example: Let $\Omega = \{(x', x_{n+1}) \in \mathbb{R}^{n+1}; x_{n+1} > 0\}$; let $G(x', x_{n+1}) = c \max\{x_{n+1}, 0\}$ for any positive constant c; and set $h = c$ on $\partial\Omega$. Then G is a Green's function for Ω with pole at ∞ and h is an associated Poisson kernel, as the following application of the divergence theorem shows:

$$\begin{aligned}
\int_{\mathbb{R}^{n+1}} G\Delta\phi \, dx &= \int_{\{x_{n+1}>0\}} c x_{n+1} \mathbf{div}\left(\nabla\phi\right) \, dx \\
&= \int_{\{x_{n+1}>0\}} \mathbf{div}\left(c x_{n+1}\nabla\phi\right) \, dx - \int_{\{x_{n+1}>0\}} \langle \nabla c x_{n+1}, \nabla\phi\rangle \, dx \\
&= \int_{\partial\{x_{n+1}>0\}} \langle c x_{n+1}\nabla\phi, \nu\rangle \, d\mathcal{H}^n - \int_{\{x_{n+1}>0\}} \langle c\nabla x_{n+1}, \nabla\phi\rangle \, dx \\
&= 0 - \int_{\{x_{n+1}>0\}} \mathbf{div}\left(\phi\nabla c x_{n+1}\right) \, dx + \int_{\{x_{n+1}>0\}} \phi\mathbf{div}\left(c\nabla x_{n+1}\right) \, dx \\
&= -\int_{\partial\{x_{n+1}>0\}} \langle \phi\nabla c x_{n+1}, \nu\rangle \, dx + \int_{\{x_{n+1}>0\}} \phi c\Delta x_{n+1} \, dx \\
&= -\int_{\partial\{x_{n+1}>0\}} \langle \phi\nabla c x_{n+1}, -e_{n+1}\rangle \, dx + 0 \\
&= \int_{\partial\{x_{n+1}>0\}} \phi h \, d\mathcal{H}^n.
\end{aligned}$$

In the surface integrals above, ν represents an outward-pointing unit normal vector on the stated boundary. □

Unbounded NTA domains always admit Green's functions with pole at ∞ – they are constructed as scaled limits of Green's functions with finite poles that tend to infinity (see [11]).

As the previous example shows, unlike Green's functions on bounded domains, Green's functions with pole at ∞ are not unique; however, on unbounded NTA domains they differ only by scalar multiplication: G and $\tilde{G}$ are Green's functions with pole at ∞ for an unbounded NTA domain Ω if and only if $G = c\tilde{G}$ for some $c > 0$. Moreover, the Poisson kernel is unique up to a set of $\mathcal{H}^n$-measure zero for a given Green's function whenever it exists, as it does when Ω is Ahlfors regular. For these reasons, authors sometimes refer to these objects as *the* Green's function and *the* Poisson kernel. See [11] for full details of the existence and uniqueness up to scalar multiplication for unbounded, Ahlfors regular NTA domains.

Chapter 2

AN UNBOUNDED GENERALIZATION OF LEWIS AND VOGEL

In this chapter, we begin with a hypothesis that says the boundary of a domain is 'flat' at one point at all large scales, and we then prove that that boundary is 'flat' everywhere at large scales. Then the domain is assumed to be unbounded and NTA and to admit a positive harmonic function that vanishes linearly at the boundary, and we show that the boundary is also 'flat' at all small scales.

2.1 Definitions

We define the following quantity to measure how much the boundary of a domain Ω differs from a plane near a point Q:

Definition 21. *Let*

$$\Theta(Q,r) = \frac{1}{r} \inf_{L \ni Q} D[\partial\Omega \cap B_r(Q), L \cap B_r(Q)],$$

the infemum being taken over all planes L containing Q.

When $\Theta(Q,r)$ is small, we think of $\partial\Omega$ as being 'flat' at Q at the scale r.

There is another notion of 'flatness' that will allow us to use the theory of partial differential equations to make estimates. It requires us to use properties of a Green's function for the domain Ω. The idea is due to Alt and Caffarelli (see [2]).

Definition 22. *For* $\Omega \subset \mathbb{R}^{n+1}$*,* $Q_0 \in \partial\Omega$*,* $\rho > 0$ *and* $\sigma_+, \sigma_-, \tau > 0$*, we say that a Green's function* G *on* Ω *satisfies*

$$G \in F(\sigma_+; \sigma_-; \tau) \text{ in } B_\rho(Q_0) \text{ in the direction } \nu$$

if

$$G(X) = 0 \text{ for } X \in B_\rho(Q) \text{ with } \langle X - Q_0, \nu\rangle \geq \sigma_+\rho, \tag{2.1}$$

$$G(X) \geq -h(Q_0)[\langle X - Q_0, \nu\rangle + \sigma_-\rho] \text{ for } X \in B_\rho(Q) \text{ with } \langle X - Q_0; \nu\rangle \leq -\sigma_-\rho, \tag{2.2}$$

and

$$\limsup_{X \to Q \in \partial\Omega \cap B_\rho(Q_0)} |\nabla G(X)| \leq (1 + \tau) \text{ and } h(Q) > (1 - \tau) \text{ for } Q \in \partial\Omega \cap B_\rho(Q_0) \tag{2.3}$$

for an associated Poisson kernel h*.*

We will refer to this notion later as **F-flatness** when we wish to distinguish it from the concept in definition 21, which we will call Θ**-flatness.**

It is immediate that, if $G \in F(\sigma_+; \sigma_-; \tau)$ in the ball $B_\rho(Q_0)$ in the direction ν, then $G \in F(2\sigma_+; 2\sigma_-; \tau)$ in the ball $B_{\frac{\rho}{2}}(Q_0)$ in the direction ν. The content of several of the lemmas later in this discussion will be refinements of this statement when G has certain properties. The relationship between the two notions of 'flatness' discussed here is illustrated in Figure 1.

If L is the plane through Q normal to ν and $G \in F(\sigma_+; \sigma_-; \tau)$ in $B_\rho(Q)$ in the direction ν, then $\Theta(Q, \rho) \leq \max\{\sigma_+; \sigma_-\}$. Going from Θ-flatness to F-flatness however requires an additional separation property.

Definition 23. *Let* $\Omega \subset \mathbb{R}^{n+1}$ *be an unbounded domain. We say that* Ω *has the* **exterior separation property at large scales** *if there exists* $R > 0$ *such that for each* $r > R$ *and* $Q \in \partial\Omega$ *there is* n*-dimensional plane* $L(Q, r)$ *containing* Q *and a choice of unit vector* $e(Q, r)$*, normal to* $L(Q, r)$*, satisfying*

$$\left\{X = (x, t) = x + te(Q, r) \in B(Q, r) : x \in L(Q, r), t > \frac{1}{4}r\right\} \subset \Omega^c. \tag{2.4}$$

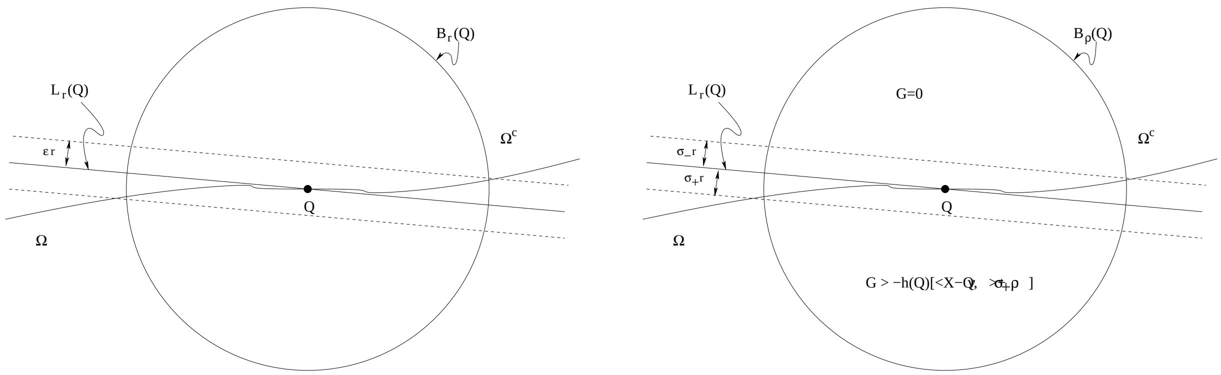

Figure 2.1: In the first graphic, $\Theta(Q, r) = \epsilon$. In the second graphic, $G \in F(\sigma_+; \sigma_-; \tau)$.

The coefficient $\frac{1}{4}$ in this definition is not special: any coefficient in the open interval $(0, 1)$ would serve our purposes, but we fix it here for convenience.

Proposition 1. *Let $\Omega \subset \mathbb{R}^{n+1}$ satisfy the exterior separation property at large scales. Let v be a Green's function on Ω with $\limsup_{X \to Q \in \partial\Omega} |\nabla v| \leq 1 + \tau$, and suppose the associated Poisson kernel satisfies $h > 1 - \tau$. Assume also that there is a hyperplane L through $Q \in \partial\Omega$ satisfying*

$$D[L \cap B_r(Q), \partial\Omega \cap B_r(Q)] < \delta r$$

for some $\delta < \frac{1}{4}$ and r sufficiently large. Then $v \in F(\delta; 1; \tau)$ in $B_r(Q)$ in the direction ν, where ν is perpendicular to L.

Proof: Let ν be a unit vector perpendicular to L. Because $D[L \cap B_r(Q), \partial\Omega \cap B_r(Q)] < \delta r$, we see that

$$\partial\Omega \cap B_r(Q) \subset \{X \in B_r(Q); -\delta r \leq \langle X - Q, \nu \rangle \leq \delta r\}.$$

Let $L(Q, r)$ be the plane and $e(Q, r)$ the unit vector guaranteed by the exterior separation property. Then we see that

$$\{X \in B_r(Q); |\langle X - Q, \nu \rangle| \geq \delta r\}$$

and

$$\left\{X = x + te(Q,r) \in B(r,Q); x \in L(Q,r) \text{ and } t \geq \frac{1}{4}r\right\}$$

have a nonempty intersection. Let Y be a point in this intersection, and if it turns out that $\langle Y - Q, \nu\rangle < 0$, replace ν by $-\nu$; thus by connectivity we have

$$\{X \in B_r(Q); \langle X - Q, \nu\rangle \geq \delta r\} \subset \Omega^c.$$

Consequently, the Green's Function for Ω is zero on this set, which tells us that (2.1) is satisfied with $\sigma^+ = \delta$. The condition (2.2) is vacuously true when $\sigma^- = 1$. The other hypotheses of this proposition guarantee the conditions in (2.3) are met, so we are finished. □

2.2 Lemmas

Lemma 4. *Suppose that* $0 \in \partial\Omega$ *and*

$$\liminf_{r\to\infty} \Theta(0,r) < \delta$$

for some sufficiently small $\delta > 0$*. Then for all* $Q \in \partial\Omega$ *we have*

$$\liminf_{r\to\infty} \Theta(Q,r) < 16\delta.$$

Proof: The hypothesis tells us that there is an increasing sequence $r_j \nearrow \infty$ such that

$$\Theta(0, r_j) < \delta.$$

Fix $Q \in \partial\Omega$ and select $J \in \mathbb{N}$ such that $r_j > 2|Q|$ for $j \geq J$. Let L_j be a plane through 0 with unit normal n_j and such that

$$\frac{1}{r_j} D[\partial\Omega \cap B_{r_j}(0); L_j \cap B_{r_j}(0)] < \delta.$$

Let

$$L_j^Q = L_j + e_Q$$

where $e_Q = \langle Q, n_j \rangle n_j$. Note that $L_j + e_Q = L_j + Q$ and $|e_Q| < \delta r_j$.

We will show that this plane satisfies the required estimates involving Hausdorff distance. There are two steps to that argument.

Step 1: First we show that $L_j^Q \cap B_{r_j/2}(Q)$ is not too far from $\partial\Omega \cap B_{r_j/2}(Q)$.

Let $X \in L_j^Q \cap B_{r_j/2}(Q)$. Then there exists $X' \in L_j^Q \cap B_{\frac{(1-4\delta)r_j}{2}}(Q)$ such that $|X - X'| < 2\delta r_j$.

Set $X'' = X' - e_Q$, so that $X'' \in L_j$. Also,

$$
\begin{aligned}
|X''| &\leq |X'' - Q| + |Q| \\
&\leq |X'' - X'| + |X' - Q| + |Q| \\
&< \delta r_j + \frac{(1-4\delta)r_j}{2} + \frac{r_j}{2} \\
&< r_j.
\end{aligned}
$$

Thus $X'' \in L \cap B_{r_j}(0)$. Hence there exists $Y \in \partial\Omega \cap B_{r_j}(0)$ such that

$$|X'' - Y| < \delta r_j.$$

Therefore

$$
\begin{aligned}
|Y - X| &\leq |Y - X''| + |X'' - X'| + |X' - X| \\
&< \delta r_j + \delta r_j + 2\delta r_j \\
&= 4\delta r_j,
\end{aligned}
$$

and

$$
\begin{aligned}
|Y - Q| &\leq |Y - X''| + |X'' - X'| + |X' - Q| \\
&< \delta r_j + \delta r_j + \frac{(1-4\delta)r_j}{2} \\
&= \frac{r_j}{2}.
\end{aligned}
$$

That is to say,

$$Y \in \partial\Omega \cap B_{r_j/2}(Q)$$

and

$$|Y - X| < 4\delta r_j.$$

This proves that

$$\sup_{X \in L_j^Q \cap B_{r_j/2}(Q)} dist(X, \partial\Omega \cap B_{r_j/2}(Q)) < 4\delta r_j.$$

Step 2: Next we show that $\partial\Omega \cap B_{r_j/2}(Q)$ is not too far from $L_j^Q \cap B_{r_j/2}(Q)$.

Let $X \in \partial\Omega \cap B_{r_j/2}(Q)$. Then there exists $Y \in L \cap B_{r_j}(0)$ such that

$$|X - Y| < \delta r_j.$$

Let $Y' = Y + e_Q$. Then $Y' \in L_j^Q \cap B_{r_j}(0)$ and

$$\begin{aligned} |X - Y'| &\leq |X - Y| + |Y - Y'| \\ &< \delta r_j + \delta r_j \\ &= 2\delta r_j. \end{aligned}$$

Note that

$$\begin{aligned} |Y' - Q| &\leq |Y' - X| + |X - Q| \\ &< 2\delta r_j + \frac{r_j}{2}. \end{aligned}$$

Then there exists $Y'' \in L_j^Q \cap B_{r_j/2}(Q)$ such that

$$|Y'' - Q| < \frac{r_j}{2}$$

and

$$|Y'' - Y'| < 2\delta r_j.$$

Hence

$$\begin{aligned}|Y'' - X| &\leq |Y'' - Y'| + |Y' - X| \\ &< 2\delta r_j + 2\delta r_j \\ &= 4\delta r_j.\end{aligned}$$

That is to say,

$$Y'' \in L_j^Q \cap B_{r_j/2}(Q)$$

and

$$|Y'' - X| < 4\delta r_j.$$

This proves that

$$\sup_{X \in \partial\Omega \cap B_{r_j/2}(Q)} dist(X, L_j^Q \cap B_{r_j/2}(Q)) < 4\delta r_j,$$

completing Step 2. Putting together the results of Step 1 and Step 2 yields

$$D[\partial\Omega \cap B_{r_j/2}(Q); L_j^Q \cap B_{r_j/2}(Q)] < 8\delta r_j.$$

Consequently,

$$\frac{1}{(r_j/2)} D[\partial\Omega \cap B_{r_j/2}(Q); L_j^Q \cap B_{r_j/2}(Q)] < 16\delta.$$

This holds for all j such that $r_j > 2|Q|$, and $Q \in \partial\Omega$ was chosen arbitrarily. Therefore, for all $Q \in \partial\Omega$, we have

$$\liminf_{r \to \infty} \Theta(Q, r) < 16\delta.$$

□

Lemma 5. *Let $\Omega \subset \mathbb{R}^{n+1}$ be unbounded and NTA, let v be a Green's function for Ω with pole at ∞, and suppose that the corresponding Poisson kernel h satisfies $h > 1 - \tau$. Let $Z \in \partial\Omega$ and assume there exists a ball $B \subset \Omega^c$ so that $Z \in \partial\Omega \cap \partial B$. Then*

$$\limsup_{X \to Z, X \in \Omega} \frac{v(X)}{dist(X, B)} \geq 1 - \tau.$$

Proof: Let

$$l = \limsup_{X \to Z, X \in \Omega} \frac{v(X)}{dist(X, B)}.$$

There exists a sequence $Y_k \in \Omega$ such that $Y_k \to Z$ and $\frac{v(Y_k)}{dist(Y_k,B)} \to l$ as $k \to \infty$. let $d_k = dist(Y_k, B)$. Then there exists $X_k \in \partial\Omega$ such that $d_k = |Y_k - X_k|$. Define

$$v_k(X) = \frac{v(d_k X + X_k)}{d_k}$$

for $X \in B_2(0)$. Define $Z_k = \frac{Y_k - X_k}{2}$.

By passing to a subsequence, we may assume that as $k \to \infty$ we have

$$Z_k \to e \quad \text{with } |e| = 1;$$

$$v_k \to v_\infty \quad \text{in } C^{0,\beta}_{loc}(\mathbb{R}^{n+1});$$

$$\nabla V_k \to \nabla v_\infty \quad \text{weakly star in } L^1_{loc}(\mathbb{R}^{n+1}) \text{ and weakly in } L^2_{loc}(\mathbb{R}^{n+1});$$

$$\frac{1}{d_k}(\partial\Omega - X_k) = \partial\{v_k > 0\} \to \partial\{v_\infty > 0\}$$

(in the Hausdorff distance sense, uniformly on compact sets); and

$$\chi_{\{v_k > 0\}} \to \chi_{\{v_\infty > 0\}} \quad \text{in } L^1_{loc}(\mathbb{R}^{n+1}).$$

Note that $v_k(Z_k) = \frac{v(Y_k)}{d_k}$, so $v_k(Z_k) \to l$ as $k \to \infty$. Also, because v_k converges uniformly to v_∞ on $B_2(0)$, we obtain $v_\infty(e) = l$.

Our goal is to show that $\Omega_\infty = \{v_\infty > 0\}$ is a half space, and that v is linear. Let r be the radius of the ball B.

Let L_k be the tangent plane to B through X_k, and let $\alpha_k = D[\partial B_{d_k}(X_k) \cap \partial B, L_k \cap B]$. Observe that $\alpha_k = 2\frac{d_k^2}{r}$. Fix

$$P_k \in \left\{ P \in B_{d_k}(X_k); \left\langle P - X_k, \frac{Y_k - X_k}{d_k} \right\rangle < -\alpha_k \right\}.$$

If $Q_k = \frac{P_k - X_k}{d_k}$, then $Q_k \in \left\{ Q \in B_2(0); \langle Q, Z_k \rangle < -\frac{d_k}{r} \right\}$, and $v_k(Q_k) \le 0$. Passing to the limit as $k \to \infty$, we conclude that if $Y \in B_2(0)$ and $\langle Y, e \rangle \le 0$ then $v_\infty(Y) = 0$.

Let $Y \in B_2(0)$ satisfy $\langle Y, Z_k\rangle > 0$; then either $d_kY + X_k \in \Omega^c$ and $V_k(Y) = 0$ or $d_kY + X_k \in \Omega$ and, given $\epsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that for $k \geq k_0$,

$$\frac{v(d_kY + X_k)}{dist(d_kY + X_k, B)} \leq l + \epsilon$$

and

$$\begin{aligned} v(d_kY + X_k) &\leq (l+\epsilon)dist(d_kY + X_k, B) \\ &\leq (l+\epsilon)\left\{\left\langle d_kY, \frac{Y_k - X_k}{d_k}\right\rangle + 2\frac{d_k^2}{r}\right\} \\ &\leq (l+\epsilon)d_k\left\{\langle Y, Z_k\rangle + 2\frac{d_k}{r}\right\}, \end{aligned}$$

which implies

$$v_k(Y) = \frac{v_k(d_kY + X_k)}{d_k} \leq (l+\epsilon)\left\{\langle Y, Z_k\rangle + 2\frac{d_k}{r}\right\}.$$

Letting $k \to \infty$, we see, for $Y \in B_2(0)$ with $\langle Y, e\rangle \geq 0$, that $v_\infty(Y) \leq (l+\epsilon)\langle Y, e\rangle$ for every $\epsilon > 0$; thus $v_\infty(Y) \leq l\langle Y, e\rangle$. Moreover, $v_\infty(e) = l$. The maximum principle guarantees that $v_\infty(Y) = l \max\{\langle Y, e\rangle, 0\}$ for all $Y \in B(0, 1)$.

If $h_k(X) = h(d_kX + X_k)$, for $\eta \in C_c^\infty(B_1(0))$, $\eta > 0$, then as $k \to \infty$ we have

$$\int_{\partial\{v_k>0\}} \eta h_k d\mathcal{H}^n = \int_{\mathbb{R}^{n+1}} \nabla v_k \cdot \nabla\eta \to -\int_{\mathbb{R}^{n+1}} \nabla v_\infty \cdot \eta = \int_{\{\langle Y,e\rangle=0\}} l\eta d\mathcal{H}^n,$$

thus

$$\lim_{k\to\infty} \int_{\partial\{v_k>0\}} \eta h_k d\mathcal{H}^n = \int_{\{\langle Y,e\rangle=0\}} l\eta d\mathcal{H}^n. \qquad (2.5)$$

On the other hand, the divergence theorem gives us

$$\int_{\partial\{v_k>0\}} \eta d\mathcal{H}^n \geq \int_{\partial\{v_k>0\}} \eta e \cdot \nu_k d\mathcal{H}^n = \int_{\{v_k>0\}} \mathrm{div}(\eta e).$$

As $k \to \infty$, we get

$$\begin{aligned} \int_{\{v_k>0\}} \mathrm{div}(\eta e) &\to \int_{\{v_\infty>0\}} \mathrm{div}(\eta e) \\ &= \int_{\partial\{v_\infty>0\}} \eta d\mathcal{H}^n \\ &= \int_{\{\langle Y,e\rangle=0\}} \eta d\mathcal{H}^n. \end{aligned}$$

Therefore

$$\liminf_{k\to\infty} \int_{\partial\{v_k>0\}} \eta d\mathcal{H}^n \geq \int_{\{\langle Y,e\rangle=0\}} \eta d\mathcal{H}^n.$$

Then because the Poisson kernel h is at least $1-\tau$ for a.e. $Q \in \partial\Omega$, we obtain

$$\lim_{k\to\infty} \int_{\partial\{v_k>0\}} h_k \eta d\mathcal{H}^n \geq (1-\tau) \lim_{k\to\infty} \int_{\partial\{v_k>0\}} \eta d\mathcal{H}^n,$$

and together with (2.5) this implies

$$l \int_{\{\langle Y,e\rangle=0\}} \eta d\mathcal{H}^n \geq (1-\tau) \int_{\{\langle Y,e\rangle=0\}} \eta d\mathcal{H}^n$$

for any $\eta \in C_c^\infty(B_1(0))$. Therefore

$$l \geq 1-\tau.$$

□

Lemma 6. *Let $\Omega \subset \mathbb{R}^{n+1}$ be unbounded and NTA; let v be a Green's function for Ω with pole at ∞. There exist $\delta_n > 0$ and τ_n depending only on n so that for $\delta \in (0,\delta_n)$ and $\tau \in (0,\tau_n)$, if $v \in F(\sigma;1;\tau)$ in $B_\rho(Q_0)$ in the direction ν, then $v \in F(2\sigma; C\sigma; \tau)$ in $B_{\frac{\rho}{2}}(Q_0)$ in the direction ν. The constant C here depends only on n.*

Proof:

Without loss of generality, assume $Q_0 = 0 \in \partial\Omega$, $\rho = 1$ and $\nu = e_{n+1}$. Define $\eta : \mathbb{R}^n \to \mathbb{R}$ by

$$\eta(y) = \begin{cases} \exp\left(\frac{-9|y|^2}{1-9|y^2}\right) & \text{for } |y| < 3 \\ 0 & \text{otherwise} \end{cases}.$$

(The precise choice of test function here isn't important – we just need to fix one since some constants will depend upon it.) Let

$$D = \{X \in B_1(0); x_{n+1} < 2\sigma - s\eta(\overline{x})\},$$

where $X = (\overline{x}, x_{n+1}) \in \mathbb{R}^{n+1}$. Choose s_0 to be the maximum s so that

$$B_1(0) \cap \{v \geq 0\} \subset D.$$

Since $0 \in \partial\Omega = \partial\{v > 0\}$, we see that $2\sigma - s_0 \geq 0$, so $s_0 \leq 2\sigma$.

Since $v \in F(\sigma; 1; \tau)$ in $B_1(0)$ in the direction e_{n+1}, there exists $Z \in \partial D \cap \partial\Omega \cap B_{\frac{1}{3}}(0)$. Note that $\partial D \cap B_1(0)$ is smooth. Let $B \subset D^c$ be a tangent ball to D at Z; because of our chose of η and the fact $s_0 \leq 2\sigma \leq 2\sigma_n$, which is small, we may take the radius of B to be a constant C_n.

Define a function V by

$$D = \begin{cases} \Delta V = 0 & \text{in } D \\ V = 0 & \text{on } \partial D \cap B_1(0) \\ V = 2\sigma - x_{n+1} & \text{on } \partial D \setminus B_1(0) \end{cases} .$$

By the maximum principle, $V > 0$ in D. Also, we see that $v \leq V$ on ∂D because $v \in F(\sigma; 1)$ in $B_1(0)$ in the direction e_{n+1}. Thus the maximum principle also tells us $v \leq V$ in D, since v is subharmonic. We also have

$$\limsup_{X \to Z, X \in \Omega} \frac{v(X)}{dist(X, B)} \leq \frac{\partial V}{\partial \vec{n}}(Z), \tag{2.6}$$

where $\vec{n}$ denotes the *inward* unit normal vector to ∂D.

For $X \in D$ define $F(X) = (2\sigma - x_{n+1}) - V(X)$. Then F is harmonic on D, continuous on $\overline{D}$ and $0 \leq F \leq s_0$ on ∂D; hence the maximum principle says $0 \leq F \leq s_0$ on D.

Since Z is a smooth point of ∂D, standard boundary regularity arguments (see section 6.2 of [6]) ensure that

$$\sup_{X \in D} |\nabla F(X)| \leq C \sup_{D} |F| \leq C s_0 \leq C\sigma.$$

Therefore

$$\frac{\partial V}{\partial x_{n+1}}(Z) = -1 - \frac{\partial F}{\partial x_{n+1}}(Z) \geq -(1 + C\sigma).$$

Consequently,

$$\begin{aligned}\frac{\partial V}{\partial \vec{n}}(Z) &= \langle \nabla V(Z), \vec{n}\rangle \\ &= \langle \nabla V(Z), \vec{n} + e_{n+1}\rangle - \frac{\partial V}{\partial x_{n+1}} \\ &\leq |\nabla V(Z)||\vec{n} + e_{n+1}| + (1 + C\sigma) \\ &\leq (1 + C\sigma)|\vec{n} + e_{n+1}| + (1 + C\sigma).\end{aligned}$$

Near Z, ∂D is a graph of a vertical translation of η, so we can calculate the inward normal vector there:

$$\vec{n}(Z) = \left(\frac{s\nabla\eta(\overline{x})}{\sqrt{1 + s^2|\nabla\eta(\overline{x})|^2}}, \frac{-1}{\sqrt{1 + s^2|\nabla\eta(\overline{x})|^2}} \right).$$

This yields $|\vec{n} + e_{n+1}| \leq C\sigma$, with $C = C(n)$.

Putting this together with (2.6), we have

$$\limsup_{X\to Z, X\in\Omega} \frac{v(X)}{dist(X,B)} \leq \frac{\partial V}{\partial \vec{n}}(Z) \leq 1 + C\sigma.$$

Together with Lemma 5, this gives us

$$1 - \tau \leq \limsup_{X\to Z, X\in\Omega} \frac{v(X)}{dist(X,B)} \leq 1 + C\sigma.$$

Next let $\xi \in \partial B_{\frac{3}{4}}(0) \cap \{x_{n+1} < -\frac{1}{2}\}$, and let ω_ξ satisfy the equation

$$\begin{cases} \Delta\omega_\xi = 0 & \text{in } D \setminus B_{\frac{1}{8}}(\xi) \\ \omega_\xi = 0 & \text{on } \partial D \\ \omega_\xi = -x_{n+1} & \text{on } \partial B_{\frac{1}{8}}(\xi) \end{cases}.$$

The Hopf boundary point lemma (see Lemma 3.4 in [6]) ensures that for some $\tilde{C} = \tilde{C}(n)$,

$$\frac{\partial \omega_\xi}{\partial \vec{n}}(Z) \geq \tilde{C} > 0.$$

Suppose $d > 0$ and that for every $X \in \overline{B}_{\frac{1}{8}}(\xi)$ we get

$$v(X) \leq V(X) + \sigma d x_{n+1}.$$

Then by the maximum principle, we have

$$v(X) \le V(X) - \sigma d \omega_\xi(X) \quad \text{on } D \setminus B_{\frac{1}{8}}(\xi).$$

Consequently,

$$1 - \tau \le \frac{\partial V}{\partial \vec{n}}(Z) - \sigma d \frac{\partial \omega_\xi}{\partial \vec{n}}(Z) \le 1 + C\sigma - \tilde{C}\sigma d.$$

Therefore

$$-\tau \le C\sigma - \tilde{C}\sigma d,$$

so

$$\tilde{C}\sigma d \le C\sigma + \tau,$$

or

$$d \le \frac{C}{\tilde{C}} + \frac{\tau}{\sigma}.$$

That is to say, if $d > \frac{C}{\tilde{C}} + \frac{\tau}{\sigma}$, then there exists $X_\xi \in \overline{B}_{\frac{1}{8}}(\xi)$ for which

$$v(X_\xi) \ge V(X_\xi) + \sigma d \left(X_\xi\right)_{n+1}.$$

Let $X \in B_{\frac{1}{4}}(X_\xi)$, and recall that $V(X) \ge -x_{n+1}$. Then

$$\begin{aligned} v(X) &\ge v(X_\xi) - \sup_{B_{\frac{1}{4}}(\xi)} |\nabla v||X - X_\xi| \\ &\ge V(X_\xi) + \sigma d (X_\xi)_{n+1} - \frac{1}{4}(1+\tau) \\ &\ge -(X_\xi)_{n+1} + \sigma d (X_\xi)_{n+1} - \frac{1}{4} - \frac{\tau}{4} \\ &\ge \frac{3}{8} - \frac{7}{8}\sigma d - \frac{1}{4} - \frac{\tau}{4} \\ &= \frac{1}{8} - \frac{7}{8}\sigma d - \frac{\tau}{4} \\ &\ge \frac{1}{8} - \frac{7}{8}\left(\sigma \frac{C}{C_n} + \tau\right) - \frac{\tau}{4}. \end{aligned}$$

Thus if σ and τ are sufficiently small, we have

$$v(X) > \frac{1}{16} > 0.$$

This tells us that v is harmonic on $B_{\frac{1}{4}}(X_\xi)$, and so is $V - v$. Furthermore, we have $V - v > 0$ on $B_{\frac{1}{4}}(X_\xi) \supset B_{\frac{1}{8}}(\xi)$, so Harnack's inequality yields

$$(V - v)(\xi) \leq C_n(V - v)(X_\xi) \leq -C\sigma d(X_\xi)_{n+1} \leq C\sigma,$$

and

$$v(\xi) \geq V(\xi) - C\sigma \geq -\xi_{n+1} - C\sigma.$$

For $X \in D \cap B_{\frac{1}{2}}(0)$ there is a $\xi \in \partial B_{\frac{3}{4}}(0)$ and a $t > 0$ such that $X = \xi + te_{n+1}$. Then

$$\begin{aligned} v(X) &= v(\xi + te_{n+1}) \\ &\geq v(\xi) - t \\ &\geq -(\xi_{n+1} + t) - C\sigma. \end{aligned}$$

Since $v \in F(\sigma; 1; \tau)$ in $B_1(0)$ in the direction e_{n+1}, the last inequality proves $v \in F(2\sigma; C\sigma; \tau)$ in $B_{\frac{1}{2}}(0)$ in the direction e_{n+1}. □

Notation: For $y \in \mathbb{R}^n$, define $B'_r(y) = \{(x \in \mathbb{R}^n; |x - y| < r\}$. In particular, $B'_r = B'_r(0)$. (The point is that our usual ambient space is $\mathbb{R}^{n+1}$, and we wish to distinguish balls in that space from balls in $\mathbb{R}^n$.)

Lemma 7. *Let $\Omega \subset \mathbb{R}^{n+1}$ be unbounded and NTA, and let v be a Green's function for Ω with pole at ∞. Suppose that $Q_j \in \partial\Omega$ with $Q_j \to Q_\infty$ and $\sigma_j \searrow 0$, and that*

$$v \in F(\sigma_j; \sigma_j; \tau) \textit{ in } B_{\rho_j}(Q_j) \textit{ in the direction } \nu_j.$$

Let R_j be the rotation that maps

$$\{(x', x_{n+1}) \in \mathbb{R}^{n+1}; x_{n+1} \geq 0\} \textit{ to } \{X + t\nu_j \in \mathbb{R}^{n+1}; \langle X, \nu_j \rangle = 0 \textit{ and } t \geq 0\}.$$

Define

$$v_j(X) = \frac{1}{\rho_j} v(\rho_j R_j X + Q_j) \tag{2.7}$$

and for $y \in B_1'$

$$f_j^+(y) = \sup\{h; (y, \sigma_j h) \in \partial\{v_j > 0\}\}, \qquad f_j^-(y) = \inf\{h; (y, \sigma_j h) \in \partial\{v_j > 0\}\}.$$

Then there exists a subsequence of indices k_j *such that*

$$\limsup_{j\to\infty,\ z\to y} f_{k_j}^+(z) = \liminf_{j\to\infty,\ z\to y} f_{k_j}^-(z). \tag{2.8}$$

Let $f(y)$ *denote the function defined by (2.8). Then* f *is continuous in* B_1'*, it satisfies* $f(0) = 0$*, and* $f_{k_j}^+$ *and* $f_{k_j}^-$ *converge uniformly to* f *on compact subsets of* B_1'.

Proof: Let $D_j = \{(y, a) \in \mathbb{R}^{n+1}; (y, \sigma_j a) \in \partial\{v_j > 0\} \cap B_1\}$. Note that $0 \in D_j$. Also, because

$$v_j \in F(\sigma_j; \sigma_j; \tau) \text{ in } B_1 \text{ in the direction } \vec{e}_{n+1},$$

we see that the scalars a in the definition of D_j are in the interval $[-1, 1]$, so $D_j \subset B_2$. Furthermore, we see that $B_1' \subset \{x'; (x', x_{n+1}) \in D_j\}$. We can pass to a subsequence such that D_j converges to a set D_∞ in the Hausdorff distance sense, and $D_\infty \subset B_2$.

For $y \in B_1'$, let $\mathcal{A}_y = \left\{\{y_j\}_{j=1}^\infty \subset B_1'; \lim_{j\to\infty} y_j = y\right\}$. Define a function f on B_1' by

$$f(y) = \sup_{\{y_j\}\in\mathcal{A}_y} \limsup_{j\to\infty} f_j^+(y_j).$$

We will show that $f(y)$ is the quantity on both sides of equation (2.8).

Step 1: "f is upper-semicontinuous."

Let $z_l \in B_1'$ satisfy $z_l \to z \in B_1'$. We want to show that $\limsup_{l\to\infty} f(z_l) \le f(z)$. W.L.O.G. we may pass to a subsequence for which $\lim_{l\to\infty} f(z_l)$ exists and equals the limit superior of the original sequence. Fix $\epsilon > 0$ and choose $\{z_l^k\}_{k=1}^\infty \in \mathcal{A}_{z_l}$ such that

$$f(z_l) - \epsilon < \limsup_{k\to\infty} f_k^+(z_l^k).$$

There is a diagonal-like subsequence $\{z_{l_i}^{k_i}\}_{i=1}^{\infty}$ such that

$$z_{l_i}^{k_i} \to z \quad \text{and} \quad f_{k_i}^{+}(z_{l_i}^{k_i}) \geq f(z_{l_i}) - \epsilon.$$

Therefore

$$\liminf_{i\to\infty} f_{k_i}^{-}(z_{l_i}^{k_i}) \geq f(z).$$

Thus $f(z) > \limsup_{i\to\infty} f(z_{l_i}) - \epsilon = \lim_{i\to\infty} f(z_{l_i}) - \epsilon$, and letting $\epsilon \to 0$ gives us the desired inequality.

Step 2: "Upper-semicontinuity implies some flatness."

Fix $y \in B_1'$, and choose $y_k \in B_1'$ so that $\lim_{k\to\infty} y_k = y$ and $\lim_{k\to\infty} f_k^{+}(y_k) = f(y)$. For $x \in B_1'$, $(x, f(x)) \in D_\infty$, so for $\epsilon > 0$ there exists $\delta_\epsilon > 0$ so that, for $\delta \in (0, \delta_\epsilon)$,

$$D_\infty \cap \{(x', x_{n+1}); x' \in B'(y, 2\delta) \text{ and } x_{n+1} > f(y) + \epsilon\} = \emptyset.$$

Then because $\{D_k\}$ converges to D_∞ in the Hausdorff-distance sense on compact subsets of B', we have that for sufficiently large k,

$$D_k \cap \{(x', x_{n+1}); x' \in B_\delta'(y) \text{ and } x_{n+1} > f_k^{+}(y_k) + \epsilon\} = \emptyset.$$

That is to say, if $(x, x_{n+1}) \in B_\delta'(y_k) \times [\sigma_k f_k^{+}(y_k) + \sigma_k\epsilon, \infty)$ then $v(x', x_{n+1}) = 0$. This implies that, for some τ,

$$v_k \in F(\frac{\sigma_k \epsilon}{\delta}; 1; \tau) \text{ in } B_\delta(y_k, \sigma_k f_k^{+}(y_k)) \text{ in the direction } \vec{e}_{n+1}.$$

Now by Lemma 6 we get for k large that

$$v_k \in F(2\frac{\sigma_k \epsilon}{\delta}; C\frac{\sigma_k \epsilon}{\delta}; \tau) \text{ in } B_{\frac{\delta}{2}}(y_k, \sigma_k f_k^{+}(y_k)) \text{ in the direction } \vec{e}_{n+1}. \tag{2.9}$$

Step 3: "Flatness lets us control $\liminf f_k^{-}$."

Now we see that if $z \in B_{\frac{\delta}{4}}'(y_k)$ and $\sigma_k h < \sigma_k f_k^{+}(y_k) - C\frac{\epsilon\sigma_k}{\delta}\delta = \sigma_k f_k^{+}(y_k) - C\epsilon\sigma_k$ then

$$f_k^{-}(z) \geq f_k^{+}(y_k) - C\epsilon \text{ for } z \in B_{\frac{\delta}{4}}'(y_k). \tag{2.10}$$

Let $\{z_k\}$ be any sequence in B_1' such that $\lim_{k\to\infty} z_k = y$. There exists $k_0 \geq 1$ so that for $k \geq k_0$, $z_k \in B'_{\frac{\delta}{4}}(y_k)$, and by the inequality (2.10)

$$f_k^-(z_k) \geq f_k^+(y_k) - C\epsilon.$$

Letting $k \to \infty$, then $\epsilon \to 0$, and using the fact that $f_k^+(y_k) \to f(y)$ as $k \to \infty$, we conclude that

$$\liminf_{k\to\infty} f_k^-(z) \geq f(y).$$

Since the sequence $\{z_k\}$ tending to y was arbitrary, this gives us equation (2.8).

Note in particular that, because the constant sequence $\{y, y, y, ...\}$ also tends to y, we get

$$\begin{aligned} f(y) = \liminf_{j\to\infty, z\to y} f_{k_j}^-(z) &\leq \liminf_{j\to\infty} f_{k_j}^-(y) \leq \limsup_{j\to\infty} f_{k_j}^-(y) \\ &\leq \limsup_{j\to\infty} f_{k_j}^+(y) \leq \limsup_{j\to\infty, z\to y} f_{k_j}^+(z) = f(y), \end{aligned}$$

so

$$\lim_{j\to\infty} f_{k_j}^-(y) = f(y). \tag{2.11}$$

Similarly,

$$\lim_{j\to\infty} f_{k_j}^+(y) = f(y). \tag{2.12}$$

Step 4: "Continuity"

We need to show that f is lower semicontinuous. Let $z_l \in B_1'$ satisfy $z_l \to z \in B_1'$. We want to show that $\liminf_{l\to\infty} f(z_l) \geq f(z)$. Similar to what we did in Step 1, we pass to a subsequence for which $\lim_{l\to\infty} f(z_l)$ exists and equals the limit inferior of the original sequence. Fix $\epsilon > 0$ and choose $\{z_l^k\}_{k=1}^\infty \in \mathcal{A}_{z_l}$ such that

$$f(z_l) + \epsilon \geq \liminf_{k\to\infty} f_k^-(z_l^k).$$

(This can be achieved now because (2.8) is valid.) There is a diagonal-like subsequence $\{z_{l_i}^{k_i}\}_{i=1}^{\infty}$ such that

$$z_{l_i}^{k_i} \to z \quad \text{and} \quad f_{k_i}^{-}(z_{l_i}^{k_i}) \leq f(z_{l_i}) + \epsilon.$$

By the definition of f we have

$$f(z) \leq \liminf_{j\to\infty, \tilde{z}\to z} f_{k_j}^{-}(\tilde{z}) \leq \liminf_{i\to\infty} f_{k_i}^{-}(z_{l_i}^{k_i}).$$

Thus $f(z) \leq \liminf_{i\to\infty} f(z_{l_i}) + \epsilon = \lim_{i\to\infty} f(z_{l_i}) + \epsilon$, and letting $\epsilon \to 0$ gives us the desired inequality.

Step 5: "$f(0) = 0$"

This result is immediate because $0 \in \partial\{v_k > 0\}$ implies

$$f_k^{-}(0) \leq 0 \leq f_k^{+}(0)$$

for all k; therefore $f(0) = 0$ according to (2.8).

Step 6: "Uniform convergence on compact sets"

Here we will use a flatness condition to help argue for uniform convergence on small balls.

Let $K \subset B'$ be compact. Since f is continuous, it is uniformly continuous on compact sets K, so that given $\epsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - f(y)| < \frac{\epsilon}{4} \quad \text{for } |x - y| < \delta, \quad x, y \in K. \tag{2.13}$$

Then there exists $\delta_\epsilon > 0$ such that equation (2.9) can be written as

$$v_k \in F(4\frac{\sigma_k \epsilon}{\delta}; 2C\frac{\sigma_k \epsilon}{\delta}; \tau) \text{ in } B_\delta(y, \sigma_k f_k^{+}(y_k)) \text{ in the direction } \vec{e}_{n+1} \tag{2.14}$$

for $\delta < \delta_\epsilon$ and $k \geq k(y, \epsilon)$. By compactness, K can be covered by finitely many balls $\{B'_{\frac{\delta_l}{2}}(y_l)\}_{l=1}^{N}$ with

$$B'_{\frac{\delta_l}{2}}(y_l) \subset B', \qquad \delta_l < \delta_0$$

and such that, for $\delta < \delta_l$, (2.14) is satisfied for $y = y_l$, $k \geq k(l, \epsilon)$. For each $l = 1, ..., N$ we have

$$f(y_l) = \lim_{j\to\infty} f^+_{k_j}(y_l) = \lim_{j\to\infty} f^-_{k_j}(y_l),$$

so there exists $j_0 \geq 1$ such that, for $j \geq j_0$ and $l = 1, ..., N$,

$$|f^+_{k_j}(y_l) - f(y_l)| < \epsilon \quad \text{and} \quad |f^-_{k_j}(y_l) - f(y_l)| < \epsilon \tag{2.15}$$

and

$$v_{k_j} \in F(4\frac{\sigma_{k_j}\epsilon}{\delta}; 2C\frac{\sigma_{k_j}\epsilon}{\delta}; \tau) \text{ in } B_{\delta_l}(y_l, \sigma_{k_j} f^+_{k_j}(y_k)) \text{ in the direction } \vec{e}_{n+1}.$$

This flatness condition implies that for $z \in B_{\delta_l}(y_l, \sigma_{k_j} f^+_{k_j}(y_l))$,

$$|f^+_{k_j}(z) - f^+_{k_j}(y_l)| \leq C\epsilon. \tag{2.16}$$

Then since $K \subset \bigcup_{l=1}^N B'_{\frac{\delta_l}{2}}(y_l)$ we have for $j \geq j_0$

$$\begin{aligned}|f^+_{k_j}(z) - f(z)| &\leq |f^+_{k_j}(z) - f^+_{k_j}(y_l)| + |f^+_{k_j}(y_l) - f(y_l)| + |f(y_l) - f(z)| \\ &\leq C\epsilon + \epsilon + \frac{\epsilon}{4}.\end{aligned}$$

where l is chosen so that $z \in B'_{\frac{\delta}{2}}(y_l)$. This gives uniform convergence of the sequence $\{f^+_{k_j}\}$, and the calculation for $\{f^-_{k_j}\}$ is the same. □

Lemma 8. *Let v satisfy the hypotheses of Lemma 7. Also suppose that $\tau_j\sigma_j^{-2} \to 0$ as $j \to \infty$. Then the function f introduced in Lemma 7 is subharmonic in B'.*

The additional hypothesis in this lemma should never really be satisfied. It is part of the negation of the conclusion of Lemma 11 below which will be proved by contradiction; this result, and the ones that follow, are key ingredients in that argument.

To accompany the blow-up of the Green's function in (2.7), we also blow-up the Poisson kernel: define

$$h_k(Q) = h(\rho_k R_k Q + Q_k). \tag{2.17}$$

Proof: Assume, to get a contradiction, that f is not subharmonic in B'. Then there exists $y_0 \in B'$ and $\rho > 0$ so that $B'_\rho(y_0) \subset B'$ and

$$f(y_0) > ⨍_{\partial B'_\rho(y_0)} f(x)\, d\mathcal{H}^{n-1}.$$

Let

$$\epsilon_0 = \frac{1}{2}\left[f(y_0) - ⨍_{\partial B'_\rho(y_0)} f(x)\, d\mathcal{H}^{n-1} \right].$$

Let g be the solution to the Dirichlet problem

$$\begin{cases} \Delta g = 0 & \text{in } B'_\rho(y_0) \\ g = f + \epsilon_0 & \text{on } \partial B'_\rho(y_0) \end{cases}.$$

Then

$$f < g \text{ on } \partial B'_\rho(y_0),$$

$$g(y_0) = ⨍_{\partial B'_\rho(y_0)} g(x) d\mathcal{H}^{n-1} = ⨍_{\partial B'_\rho(y_0)} f(x)\, d\mathcal{H}^{n-1} + \epsilon_0,$$

$$g(y_0) = \frac{1}{2}\left\{ f(y_0) + ⨍_{\partial B'_\rho(y_0)} f(x)\, d\mathcal{H}^{n-1} \right\},$$

and

$$g(y_0) < f(y_0).$$

Summarizing, we have

$$\begin{cases} \Delta g = 0 & \text{in } B'_\rho(y_0) \\ g > f & \text{in } \partial B'_\rho(y_0) \\ g(y_0) < f(y_0) \end{cases}. \tag{2.18}$$

Let $Z = B'_\rho(y_0) \times \mathbb{R}$. For ϕ defined on $\mathbb{R}^n$ write

$$Z^+(\phi) = \{(y, h) \in Z; h > \phi(y)\}$$

$$Z^-(\phi) = \{(y, h) \in Z; h < \phi(y)\}$$

$$Z^0(\phi) = \{(y, h) \in Z; h = \phi(y)\}.$$

By adding an arbitrarily small constant γ to g if necessary, we may assume that, for k sufficiently large,

$$\mathcal{H}^n(Z^0(\sigma_k g) \cap \partial\{v_k > 0\}) = 0,$$

while g still satisfies (2.18) and

$$g(y_0) = \fint_{\partial B'_\rho(y_0)} g(x)\, d\mathcal{H}^{n-1} = \fint_{\partial B'_\rho(y_0)} f(x)\, d\mathcal{H}^{n-1} + \epsilon_0 + \gamma.$$

Claim 1: For k sufficiently large,

$$\mathcal{H}^n(Z^+(\sigma_k g) \cap \partial\{v_k > 0\}) \le \frac{1+\tau_k}{1-\tau_k}\mathcal{H}^n(Z^0(\sigma_k g) \cap \{v_k > 0\}).$$

Claim 2: Let $E_k = \{v_k > 0\} \cap Z^-(\sigma_k g)$. E_k is a set of locally finite perimeter and

$$\mathcal{H}^n(Z \cap \partial^* E_k) \le \mathcal{H}^n(\partial\{v_k > 0\} \cap Z^+(\sigma_k q)) + \mathcal{H}^n(\{v_k = 0\} \cap Z^0(\sigma_k g)).$$

Here $\partial^* E_k$ denotes the reduced boundary of E_k.

Claim 3: There exists a constant $C > 0$ such that

$$\mathcal{H}^n(Z \cap \partial^* E_k) \ge \mathcal{H}^n(Z^0(\sigma_k g)) + C\sigma_k^2\rho^n.$$

We can use these three claims to obtain the desired contradiction as follows.

$$\begin{aligned}
\mathcal{H}^n(Z^0(\sigma_k g)) + C\sigma_k^2\rho^n &\le \mathcal{H}^n(Z \cap \partial^* E_k) \\
&\le \mathcal{H}^n(\partial\{v_k > 0\} \cap Z^+(\sigma_k q)) + \mathcal{H}^n(\{v_k = 0\} \cap Z^0(\sigma_k g)) \\
&\le \frac{1+\tau_k}{1-\tau_k}\mathcal{H}^n(Z^0(\sigma_k g) \cap \{v_k > 0\}) + \mathcal{H}^n(\{v_k = 0\} \cap Z^0(\sigma_k g)) \\
&\le \frac{2\tau_k}{1-\tau_k}\mathcal{H}^n(Z^0(\sigma_k g) \cap \{v_k > 0\}) + \mathcal{H}^n(Z^0(\sigma_k g)),
\end{aligned}$$

which implies

$$\begin{aligned}
C\sigma_k^2\rho^n &\le \frac{2\tau_k}{1-\tau_k}\mathcal{H}^n(Z^0(\sigma_k g) \cap \{v_k > 0\}) \\
&\le \frac{2\tau_k}{1-\tau_k}\int_{B'_\rho(y_0)} \sqrt{1 + \sigma_k^2|\nabla g|^2}.
\end{aligned}$$

For $\tau_k < \frac{1}{2}$ and $\sigma_k < 1$, this yields $C\sigma_k^2 \leq C'\tau_k$, which contradicts the fact that $\tau_k \sigma_k^{-2} \to 0$ as $k \to \infty$. This contradiction tells us f must be subharmonic in B'.

Proof of Claim 1: Since $h_k > 1 - \tau_k$ we see that

$$\begin{aligned}\mathcal{H}^n(Z^+(\sigma_k g) \cap \partial\{v_k > 0\}) &= \int_{Z^+(\sigma_k g)\cap\partial\{v_k>0\}} d\mathcal{H}^n \\ &\leq \frac{1}{1-\tau_k} \int_{Z^+(\sigma_k g)\cap\partial\{v_k>0\}} h_k \, d\mathcal{H}^n.\end{aligned}$$

For $\phi \in C_c^\infty(\mathbb{R}^{n+1})$ and k large enough we have

$$-\int_{\overline{\{v_k>0\}}\cap\partial Z^+(\sigma_k g)} \nabla v_k \cdot \nu = \int_{\partial\{v_k>0\}\cap Z^+(\sigma_k g)} \phi h_k \, d\mathcal{H}^n.$$

Letting $\phi \to \chi_{Z^+(\sigma_k g)}$, this yields

$$-\int_{\overline{\{v_k>0\}}\cap\partial Z^+(\sigma_k g)} \nabla v_k \cdot \nu = \int_{\partial\{v_k>0\}\cap Z^+(\sigma_k g)} h_k \, d\mathcal{H}^n,$$

where ν denotes the outward pointing unit normal vector. Therefore

$$\begin{aligned}\mathcal{H}^n(Z^+(\sigma_k g) \cap \partial\{v_k > 0\}) &\leq \frac{1}{1-\tau_k} \int_{\{v_k>0\}\cap\partial Z^+(\sigma_k g)} |\nabla v_k| \\ &\leq \frac{1+\tau_k}{1-\tau_k} \mathcal{H}^n(\{v_k > 0\} \cap Z^0(\sigma_k g)).\end{aligned}$$

Proof of Claim 2: This follows immediately from the finite additivity of $\mathcal{H}^n$.

Proof of Claim 3: Recall that f is continuous, $f < g$ on $\partial B'_\rho(y_0)$ and $f(y_0) > g(y_0)$. For $\delta \in (0, \delta_0)$, where $\delta_0 = \frac{1}{2}(f(y_0) - g(y_0)) > 0$ there exists $\kappa_0 \in (0, \frac{\delta}{2})$ so that for $\kappa \in (0, \kappa_0)$

$$f|_{B'_{2\kappa}(y_0)} > g|_{B'_{2\kappa}(y_0)} + \delta.$$

For κ as above let $\zeta_k \in C_c^\infty(\mathbb{R}^n)$ satisfying $0 \leq \zeta_k \leq 1$, $\zeta_k = 1$ on $B'_\kappa(y_0)$, $\zeta_k = 0$ outside $B'_{2\kappa}(y_0)$ and $|\nabla \zeta_k| \leq \frac{C_n}{\kappa}$ for $l \geq 1$. Using standard elliptic PDE estimates one can show that there exists $\kappa_1 \in (0, \kappa_0)$ so that if η satisfies

$$\begin{cases} \Delta\eta = -\zeta_k & \text{in } B'_\rho(y_0) \\ \eta = g & \text{on } \partial B'_\rho(y_0) \end{cases}$$

for $\kappa \in (0, \kappa_1)$ then

$$\eta < f \text{ on } B'_{2\kappa}(y_0).$$

Moreover note that since η is superharmonic and g is harmonic on $B'_\rho(y_0)$ then

$$g \leq \eta \text{ on } B'_\rho(y_0).$$

A similar argument to the one presented above guarantees that we can choose $\kappa \in (0, \kappa_1)$ so that

$$\mathcal{H}^n(Z^0(\sigma_k \eta) \cap \partial^* E_k) = 0.$$

We estimate

$$\begin{aligned}
\mathcal{H}^n(Z^+(\sigma_k \eta) \cap \partial^* E_k) &= \int_{Z^+(\sigma_k \eta) \cap \partial^* E_k} d\mathcal{H}^n \\
&\geq \int_{Z^+(\sigma_k \eta) \cap \partial^* E_k} \nu_k \cdot \nu_{E_k} \, d\mathcal{H}^n \\
&= \int_{E_k \cap Z^+(\sigma_k \eta)} \text{div} \nu_k - \int_{\partial Z^+(\sigma_k \eta) \cap E_k} \nu_k \cdot (-\nu_k),
\end{aligned}$$

where $\nu_k(\overline{x}, x_{n+1}) = \frac{1}{\sqrt{1+\sigma_k^2 |\nabla \eta(\overline{x})|^2}}(-\sigma_k \nabla \eta(\overline{x}), 1)$ is a smooth function on Z. Note that on $Z^0(\sigma_k \eta)$, ν_k is the unit normal pointing into $Z^+(\sigma_k \eta)$. Note that

$$\begin{aligned}
\text{div}_{\mathbb{R}^{n+1}} \nu_k &= -\sigma_k \text{div}_{\mathbb{R}^n} \frac{\nabla \eta}{\sqrt{1+\sigma_k^2 |\nabla \eta|^2}} \\
&= -\sigma_k \frac{\Delta \eta}{\sqrt{1+\sigma_k^2 |\nabla \eta|^2}} + \sigma_k^3 \sum_{i=1}^n \frac{\nabla_i \eta \nabla_i \eta \nabla_i \nabla_i \eta}{(1+\sigma_k^2 |\nabla \eta|^2)^{\frac{3}{2}}},
\end{aligned}$$

since $\Delta \eta \leq 0$ and $\nabla^2 \eta$ is bounded, we get

$$\text{div}_{\mathbb{R}^{n+1}} \nu_k \geq -c\sigma_k^3.$$

Hence

$$\mathcal{H}^n(Z^+(\sigma_k \eta) \cap \partial^* E_k) \geq \mathcal{H}^n(Z^0(\sigma_k \eta) \cap E_k) - C\sigma_k^3 \mathcal{H}^{n+1}(E_k \cap Z^+(\sigma_k \eta))$$

since $g \leq \eta$ then $Z^+(\sigma_k \eta) \subset Z^+(\sigma_k g)$ and

$$E_k \cap Z^+(\sigma_k \eta) = \{v_k > 0\} \cap Z^+(\sigma_k \eta).$$

Since $v_k \in F(\sigma_k, \sigma_k, \tau_k)$ in $B_1(0)$ in the direction $\vec{e}_{n+1}$ and g is bounded, we get that

$$\begin{aligned} E_k \cap Z^+(\sigma_k \eta) &= \{v_k > 0\} \cap Z^+(\sigma_k \eta) \\ &\subset B'_\rho(y_0) \times (-\infty, \sigma_k] \cap Z^+(\sigma_k g) \\ &\subset B'_\rho(y_0) \times [-C\sigma_k, \sigma_k]. \end{aligned}$$

Therefore

$$\mathcal{H}^n(Z^+(\sigma_k \eta) \cap \partial^* E_k) \geq \mathcal{H}^n(Z^0(\sigma_k \eta) - C\sigma_k^4 \rho^n.$$

A similar argument proves that

$$\mathcal{H}^n(Z^-(\sigma_k \eta) \cap \partial^* E_k) \geq \mathcal{H}^n(Z^0(\sigma_k \eta) \setminus E_k) - C\sigma_k^3 \mathcal{H}^{n+1}(Z^-(\sigma_k \eta) \setminus E_k).$$

Note that

$$Z^-(\sigma_k \eta) \setminus E_k = Z^-(\sigma_k \eta) \cap \{v_k = 0\} \cap Z^+(\sigma_k g),$$

since $\mathcal{H}^{n+1}(Z^-(\sigma_k \eta) \cap \{v_k = 0\} \cap Z(\sigma_k \eta)) = 0$, we only need to look at the term $Z^-(\sigma_k \eta) \cap \{v_k = 0\} \cap Z^+(\sigma_k g)$. Using again the fact that $v_k \in F(\sigma_k; \sigma_k; \tau_k)$ in $B_1(0)$, we have that

$$\left(Z^-(\sigma_k \eta) \cap \{v_k = 0\} \cap Z^+(\sigma_k g)\right) \subset \left(Z^-(\sigma_k \eta) \cap Z^+(\sigma_k g) \cap B'_\rho(y_0) \times [-\sigma_k, +\infty)\right).$$

Recall that on $B_{2\kappa}(y_0)$, $\eta \leq f \leq 1$, therefore

$$Z^-(\sigma_k \eta) \cap B'_{2\kappa}(y_0) \times [-\sigma_k, +\infty) \subset B'_{2\kappa}(y_0) \times [-\sigma_k, \sigma_k].$$

On $B'_\rho(y_0) \setminus B'_{2\kappa}(y_0)$, $\Delta \eta = 0$, $\eta = g$ on $\partial B'_\rho(y_0)$ and $\eta \leq f$ on $\partial B'_{2\kappa}(y_0)$ thus η is bounded on $B'_\rho(y_0) \setminus B'_{2\kappa}(y_0)$ which guarantees that

$$\left(Z^-(\sigma_k \eta) \cap (B'_\rho(y_0) \setminus B'_{2\kappa}(y_0)) \times [-\sigma_k, \infty)\right) \subset (B'_{2\kappa}(y_0) \times [-\sigma_k, \sigma_k]).$$

Hence

$$\mathcal{H}^n(Z^-(\sigma_k \eta) \cap \partial^* E_k) \geq \mathcal{H}^n(Z^-(\sigma_k \eta) \setminus E_k) - C\sigma_k^4 \rho^n.$$

We deduce that

$$\mathcal{H}^n(Z \cap \partial^* E_k) \geq \mathcal{H}^n(Z^0(\sigma_k \eta)) - C\sigma_k^4 \rho^n.$$

We estimate

$$\begin{aligned}\mathcal{H}^n(Z^0(\sigma_k\eta)) - \mathcal{H}^n(Z^0(\sigma_k g)) &= \int_{B'_\rho(y_0)} \sqrt{1+\sigma_k^2|\nabla\eta|^2} - \sqrt{1+\sigma_k^2|\nabla g|^2} \\ &= \int_{B'_\rho(y_0)} \sqrt{1+\sigma_k^2|\nabla g|^2 + \sigma_k^2|\nabla\eta - \nabla g|^2 + 2\sigma_k^2\langle\nabla\eta - \nabla g, \nabla g\rangle} \\ &\qquad - \sqrt{1+\sigma_k^2|\nabla g|^2}.\end{aligned}$$

Using Taylor's expansion, the fact that g is harmonic in $B'_\rho(y_0)$, $g = \eta$ on $\partial B'_\rho(y_0)$ and Poincare's inequality we get that for k large enough

$$\begin{aligned}\mathcal{H}^n(Z^0(\sigma_k\eta)) - \mathcal{H}^n(Z^0(\sigma_k g)) &\geq \sigma_k^2 \int_{B'_\rho(y_0)} |\nabla\eta - \nabla g|^2 + 2\sigma_k^2 \int_{B'_\rho(y_0)} \langle\nabla(\eta - g), \nabla g\rangle - C\sigma_k^4\rho^n \\ &\geq \sigma_k^2 \int_{B'_\rho(y_0)} |\nabla\eta - \nabla g|^2 - C\sigma_k^4\rho^n \\ &\geq \sigma_k^2\rho^{-2} \int_{B'_\rho(y_0)} |\eta - g|^2 - C\sigma_k^4\rho^n \\ &\geq C\sigma_k^2\rho^n.\end{aligned}$$

□

Lemma 9. *There is a constant $C = C(n) > 0$ such that, for $y \in B'_{\frac{1}{2}}$, the function f from Lemma 8 satisfies*

$$0 \leq \int_0^{\frac{1}{4}} \frac{1}{r^2}(f_{y,r} - f(y))dr \leq C,$$

where

$$f_{y,r} = \fint_{\partial B'_r(y)} f \, d\mathcal{H}^{n-1}.$$

Proof: Without loss of generality, we may assume $y = 0$. Since $f(0) = 0$ it is enough to show that

$$0 \leq \int_0^{\frac{1}{4}} \frac{1}{r^2} \fint_{\partial B'_r} f \, d\mathcal{H}^{n-1} \leq C,$$

where C depends only on n. Because f is subharmonic,

$$0 = f(0) \leq \fint_{\partial B'_r} f \, d\mathcal{H}^{n-1},$$

which proves the first inequality.

Let $p > 2\sigma_j$ be small and let G_p denote the Green's function of $B_{\frac{1}{2}}(0) \cap \{x_{n+1} < 0\}$ with pole $-pe_{n+1}$. By reflection G_p can be extended to a smooth function on $B_{\frac{1}{2}}(0) \setminus \{\pm pe_{n+1}\}$ with $G_p(\overline{x}, x_{n+1}) = -G(\overline{x}, -x_{n+1})$ for $x_{n+1} > 0$. For j large let $G_p^j(X) = G_p(X + \sigma_j e_{n+1})$ be defined on $B_{\frac{1}{2}}(-\sigma_j e_{n+1}) \setminus \{(\sigma_j \pm p)e_{n+1}\}$. We denote by $B_{\frac{1}{2}} = B_{\frac{1}{2}}(0)$ and by $B^j_{\frac{1}{2}} = B_{\frac{1}{2}}(-\sigma_j e_{n+1})$. We may assume that $\mathcal{H}^n(\partial B^j_{\frac{1}{2}} \cap \partial\{v_j\}) = 0$. Green's formula ensures that

$$-\int_{B^j_{\frac{1}{2}}} \langle v_j, \nabla G^j_p \rangle = \int_{\partial B^j_{\frac{1}{2}}} v_j \partial_\nu G^j_p - v_j(-(p + \sigma_j)e_{n+1}),$$

where $\partial_\nu G^j_p = \langle \nabla G^j_p, \nu \rangle$, and ν denotes the inward pointing unit normal to $\partial B'_{\frac{1}{2}}$. On the other hand

$$-\int_{\partial B^j_{\frac{1}{2}}} \langle \nabla v_j, \nabla G^j_p \rangle = \int_{\partial\{v_j > 0\} \cap B^j_{\frac{1}{2}}} h_j G^j_p \, d\mathcal{H}^n.$$

Let ν_j denote the inward pointing unit normal vector to $\partial\Omega_j = \partial\{v_j > 0\}$ then by Green's formula we have

$$\int_{B'_{\frac{1}{2}} \cap \partial\{v_j > 0\}} \langle G^j_p e_{n+1} - x_{n+1} \nabla G^j_p, \nu_j \rangle \, d\mathcal{H}^n = (\sigma_j + p) + \int_{B'_{\frac{1}{2}} \cap \{v_j > 0\}} x_{n+1} \partial_\nu G^j_p.$$

Thus we obtain

$$\begin{aligned}
\int_{B^j_{\frac{1}{2}}\cap\partial\{v_j>0\}} & x_{n+1}\partial_{\nu_j} G^j_p \, d\mathcal{H}^n \\
&= \int_{B_{\frac{1}{2}}\cap\partial\{v_j>0\}} (p_j + \langle e_{n+1}, \nu_j\rangle) G^j_p \, d\mathcal{H}^n \\
&\qquad - \int_{\partial B^j_{\frac{1}{2}}\cap\{v_j>0\}} (x_{n+1}+v_j)\partial G^j_p \, d\mathcal{H}^n + G_j(-(p+\sigma_j)e_{n+1}) - (\sigma_j + p) \\
&= \int_{B_{\frac{1}{2}}\cap\partial\{v_j>0\}} \left(\frac{h_j}{1-\tau_j} + \langle e_{n+1}, \nu_j\rangle\right) G^j_p \, d\mathcal{H}^n \\
&\qquad - \tau_j \int_{B_{\frac{1}{2}}\cap\partial\{v_j\}} h_j G^j_p \, d\mathcal{H}^n + v_j(-(p+\sigma_j)e_{n+1}) - (\sigma_j + p) \\
&\qquad\quad - \int_{\partial B^j_{\frac{1}{2}}\cap\{v_j>0\}} (x_{n+1}+v_j)\partial_\nu G^j_p \, d\mathcal{H}^n \\
&= \int_{B_{\frac{1}{2}}\cap\partial\{v_j>0\}} \left(\frac{h_j}{1-\tau_j} + \langle e_{n+1}, \nu_j\rangle\right) G^j_p \, d\mathcal{H}^n + (1+\tau_j)G_j(-(p+\sigma_j)e_{n+1}) - (\sigma_j + p) \\
&\qquad - \int_{\partial B^j_{\frac{1}{2}}\cap\{v_j>0\}} (x_{n+1}+G_j(1+\tau_j))\partial_{\nu_j} G^j_p \, d\mathcal{H}^n.
\end{aligned}$$

Since $\sigma_j - p < -\sigma_j$ and $G_j \in F(\sigma_j;\sigma_j;\tau_j)$ in $B_1(0)$ in the direction e_{n+1}, then $G^j_p \le 0$ on $\partial\{v_j>0\}\cap B^j_{\frac{1}{2}}$. Furthermore, since $h_j \ge 1-\tau_j$ on $B^j_{\frac{1}{2}}\cap\partial\{v_j>0\}$, we have

$$\int_{B_{\frac{1}{2}}\cap\partial\{v_j>0\}} \left(\frac{h_j}{1-\tau_j} + \langle e_{n+1}, \nu_j\rangle\right) G^j_p \le 0.$$

Since $G_j(0)=0$, we get

$$|G_j(-(p+\sigma_j)e_{n+1})| \le \sup_{B_1(0)} |\nabla v_j|(p+\sigma_j) \le (1+\tau_j)(p+\sigma_j).$$

Hence

$$(1+\tau_j)v_j(-(p+\sigma_j)e_{n+1}) - (\sigma_j+p) \le 3\tau_j(p+\sigma_j).$$

Since $\{v_j>0\}\subset\{x_{n+1}<\sigma_j\}$, for $x_{n+1}\le\sigma_j$ we have in $B_1(0)$

$$v_j(\overline{x}, x_{n+1}) = |v_j(\overline{x}, x_{n+1} - v_j(\overline{x}, \sigma_j)| \le (1+\tau_j)(\sigma_j - x_{n+1})$$

which yields

$$x_{n+1} \leq x_{n+1} + v_j(1+\tau_j) \leq (1-(1+\tau_j)^2)x_{n+1} + (1-\tau_j)^2\sigma_j.$$

Thus

$$0 \leq x_{n+1} + (1+\tau_j)v_j \leq (1+\tau_j)^2\sigma_j \text{ for } x_{n+1} \in [0,\sigma_j],$$

and

$$-\sigma_j \leq x_{n+1} + (1+\tau_j)v_j \leq (1+\tau_j)\sigma_j \text{ for } x_{n+1} \in [-\sigma_j, 0].$$

Since $v_j \in F(\sigma_j;\sigma_j;\tau_j)$ in $B_1(0)$ in the direction e_{n+1} with $h_j(0)=1$ then

$$\begin{aligned} x_{n+1} + G_j(1+\tau_j) &\geq x_{n+1} + (1+\tau_j)(-x_{n+1}-\sigma_j) \\ &\geq -\tau_j x_{n+1} - \sigma_j(1+\tau_j) \\ &\geq -\sigma_j(1+\tau_j) \text{ for } x_{n+1} \leq -\sigma_j. \end{aligned}$$

We combine the fact that $\partial_\nu G_p^j \geq 0$ with the last results to obtain

$$-\int_{\partial B^j_{\frac{1}{2}} \cap \{v_j>0\}} (x_{n+1} + (1+\tau_j)v_j)\partial_\nu G_p^j \leq \sigma(1+\tau_j)\int_{\partial B^j_{\frac{1}{2}} \cap \{v_j>0\}\cap\{x_{n+1}<0\}} \partial_\nu G_p^j.$$

Using the fact that $\sigma_j^{-2}\tau_j \leq 1$ for j large enough, and that $1 \geq p \geq 2\sigma_j$ we conclude that

$$\frac{1}{\sigma_j}\int_{B^j_{\frac{1}{2}} \cap \partial\{v_j>0\}} x_{n+1}\partial_{\nu_j} G_p^j \leq 9\sigma_j + 2\int_{\partial B^j_{\frac{1}{2}} \cap \{v_j>0\}\cap\{x_{n+1}<0\}} \partial_\nu G_p^j.$$

Thus

$$\limsup_{j\to\infty} \frac{1}{\sigma_j}\int_{B^j_{\frac{1}{2}} \cap \partial\{v_j>0\}} x_{n+1}\partial_{\nu_j} G_p^j \leq 2\int_{\partial B_{\frac{1}{2}} \cap \{x_{n+1}\leq 0\}} \partial_\nu G_p \leq Cp.$$

Since $v_j \in F(\sigma_j,\sigma_j;\tau_j)$, we have that $\chi_{\{v_j>0\}} \to \chi_{\{x_{n+1}\leq 0\}}$ as $j\to\infty$ in $L^1_{loc}(B_1(0))$ and $\partial\{v_j>0\} = \partial\Omega_j \to \{x_{n+1}=0\}$ in the Hausdorff distance sense uniformly on compact sets. Moreover, since f_j^+ and f_j^- converge uniformly to f on compact sets and ∇G_p^j converges to G_p smoothly away from $\pm p e_{n+1}$, we have that

$$\sup_{(\overline{x},x_{n+1})\in\partial\{v_j>0\}\cap B^j_{\frac{1}{2}}} \left| \frac{x_{n+1}}{\sigma_j}\nabla G_p^j(\overline{x},x_{n+1}) - f(x)\nabla G_p(x,0)\right| \to 0 \text{ as } j\to\infty.$$

Therefore

$$\frac{1}{p}\int_{B'_{\frac{1}{2}}} f(x)\nabla_{-e_{n+1}}G_p(\overline{x},0)\,dx \leq C.$$

Note that $\nabla_{-e_{n+1}}G_p\big|_{x_{n+1}=0} = -\frac{\partial G_p}{\partial x_{n+1}}\Big|_{x_{n+1}=0}$ is radially symmetric on $B'_{\frac{1}{2}}$. In fact, by definition $G_p(\overline{x},x_{n+1}) = F(x,x_{n+1}+p) - u_p(\overline{x},x_{n+1})$ where F denotes the fundamental solution of the Laplacian in $\mathbb{R}^{n+1}$ and u_p is the unique harmonic function satisfying $\Delta u_p = 0$ in $B_{\frac{1}{2}} \cap \{x_{n+1} < 0\} = B^-_{\frac{1}{2}}$ and $v_p(\overline{x},x_{n+1}) = F(x,x_{n+1}+p)$ for $(\overline{x},x_{n+1}) \in \partial B^-_{\frac{1}{2}}$. Since $F(\overline{x},x_{n+1}+p) = \tilde{F}(|x|,|x_{n+1}+p|)$, then $u_p(\overline{x},x_{n+1}) = \tilde{u}_p(|x|,x_{n+1})$ and $G_p(\overline{x},x_{n+1}+p) = \tilde{G}_p(|x|,x_{n+1})$, which justifies the fact that $-\frac{\partial G_p}{\partial x_{n+1}}\Big|_{x_{n+1}=0}$ is radially symmetric on $B'_{\frac{1}{2}}$.

Let $g_p(r) = g_p(|x|) = -\frac{\partial G_p}{\partial x_{n+1}}(x,0)$ for $x = r\theta$ and $\theta \in \mathbb{S}^{n-1}$. With this notation,

$$\begin{aligned}\frac{1}{p}\int_{B'_{\frac{1}{2}}} f(x)g_p(|x|)\,dx &= \frac{1}{p}\int_0^{\frac{1}{2}} r^{n-1}g_p(r)\int_{\mathbb{S}^{n-1}} f(r\theta)\,d\theta\,dr\\ &= \frac{\sigma_{n-1}}{p}\int_0^{\frac{1}{2}} r^{n-1}g_p(r) ⨍_{\partial B'_r} f\,d\mathcal{H}^{n-1}\,dr\\ &\leq C.\end{aligned}$$

Comparing $g_p(r)$ with the Poisson kernel of $\mathbb{R}^{n+1}$ with pole at $-pe_{n+1}$, $P_p(r)$, and using the comparison principle for nonnegative harmonic functions on $B^-_{\frac{1}{2}}$ we obtain

$$\frac{g_p(r)}{P_p(r)} = \lim_{x\to(r\theta,0)}\frac{G_p(X)}{G_p^\infty(X)} \geq C_n\frac{G_p(A_p)}{G_p^\infty(A_p)},$$

where G_p^∞ denotes the Green's function of $\mathbb{R}^{n+1}$ with pole at $-pe_{n+1}$; and $A_p = -\frac{p}{64}e_{n+1}$. Since $G_p^\infty(A_p) \leq \frac{C_n}{p^{n-1}}$ and $G_p(A_p) \geq \frac{C_n}{p^{n-1}}$, this means

$$g_p(r) \geq C_n\frac{p}{(r^2+p^2)^{(n+1)/2}}.$$

Therefore

$$\int_0^{\frac{1}{2}} \frac{r^{n-1}}{(r^2+p^2)^{(n+1)/2}}\left(⨍_{\partial B'_r} f(x)\,dx\right)\,dr \leq C,$$

and C only depends on n. Letting p tend to 0, we are done. □

Lemma 10. *The function f in Lemma 8 is Lipschitz with a constant that only depends on n. Furthermore, there exists a large constant $C = C(n) > 0$ such that for any given $\theta \in (0,1)$ there exists $\eta = \eta(\theta) > 0$ and $l \in \mathbb{R}^n \times \{0\}$, with $|l| \leq C$ so that*

$$f(y) \leq \langle l, y \rangle + \frac{\theta}{2}\eta \quad \textit{for } y \in B'_\eta.$$

Proof: For $y \in \overline{B'_{\frac{1}{16}}}$, let G_y denote the Green's Function of $B'_{\frac{1}{8}}$ with pole at y. Since f is subharmonic in $B'_{\frac{1}{2}}$, Green's formula ensures that

$$f(y) = \int_{\partial B'_{\frac{1}{8}}} f(q) \frac{\partial G_y(q)}{\partial \nu} \, d\mathcal{H}^{n-1}(q) - \int_{B'_{\frac{1}{8}}} G_y \, d\lambda,$$

where $\lambda = \Delta f$ is a nonnegative Radon measure and ν denotes the inward-pointing unit normal to $\partial B'_{\frac{1}{8}}$. Recall that for $q \in \partial B'_{\frac{1}{8}}$,

$$\frac{\partial G_y}{\partial \nu}(q) = 8 \frac{\left(\frac{1}{8}\right)^2 - |y|^2}{n\omega_n} \cdot \frac{1}{|y-q|^n}.$$

Note that for $q \in \partial B'_{\frac{1}{8}}$, $\frac{\partial G_y}{\partial \nu}(q)$ is a smooth function of y in $\overline{B}'_{\frac{1}{16}}$. Since $|f| \leq 1$ for $x, y \in \overline{B}'_{\frac{1}{16}}$ we get that

$$\begin{aligned} |f(x) - f(y)| = & \left| \int_{\partial B'_{\frac{1}{8}}} f(q) \left\{ \frac{\partial G_y}{\partial \nu}(q) - \frac{\partial G_x}{\partial \nu}(q) \right\} \, d\mathcal{H}^{n-1}(q) \right| \\ & + \left| \int_{B'_{\frac{1}{8}}} |G_y - G_x| \, d\lambda \right| \\ & \leq C|x-y| + \int_{B'_{\frac{1}{8}}} |G_y - G_x| \, d\lambda, \end{aligned}$$

where $C > 0$ is a constant that only depends on n. In order to estimate the second term in the right hand side, note that

$$\int_{B'_{\frac{1}{8}}} |G_y - G_x| \, d\lambda \leq \left(\int_{B'_{\frac{1}{8}}} \int_0^1 |\nabla_{\xi_t} G_{\xi_t}(z)| \, dt \, d\lambda(z) \right) |x-y|$$
$$\leq C|x-y| \int_0^1 \int_{B'_{\frac{1}{8}}} \frac{d\lambda(z)}{|\xi_t - z|^{n-1}} \, dt$$

where $\xi_t = tx + (1-t)y$ for $[0,1]$.

Therefore in order to prove that f is Lipschitz on $\overline{B}'_{\frac{1}{16}}$, we need to show that there is a constant C only depending on n such that for $\xi \in \overline{B}'_{\frac{1}{16}}$,

$$\int_{B'_{\frac{1}{8}}} \frac{d\lambda(z)}{|\xi - z|^{n-1}} \leq C.$$

Let $\xi \in \overline{B}'_{\frac{1}{16}}$, and let G^r denote the Green's function of $B'_r(\xi)$ with pole ξ. By Green's formula we have that if $f_{\xi,r} = \fint_{B'_r(\xi)} f \, d\mathcal{H}^{n-1}$ then

$$f_{\xi,r} - f(\xi) = \fint_{\partial B'_r(\xi)} (f - f(\xi)) \, d\mathcal{H}^{n-1} = \int_{B'_r(\xi)} G^r(z) \, d\lambda(z),$$

because $-\left.\frac{\partial G^r(x)}{\partial \nu}\right|_{x \in \partial B'_r(\xi)} = \frac{1}{\sigma_{n-1} r^{n-1}}$. Recalling that $G^r(z) = C_n \frac{1}{|z-\xi|^{n-2}} - \frac{C_n}{r^{n-2}}$ we obtain for $n \geq 3$

$$\begin{aligned}
C &\geq \int_0^{\frac{1}{4}} \frac{1}{r^2} (f_{\xi,r} - f) \, dr \\
&= \int_0^{\frac{1}{4}} \frac{1}{r^2} \int_{B'_r(\xi)} G^r(z) \, d\lambda(z) \\
&\geq \int_{B'_{\frac{1}{4}}(\xi)} \left(\int_{|\xi - z|}^{\frac{1}{4}} \frac{1}{r^2} G^r(z) \, dr \right) d\lambda(z) \\
&\geq \int_{B'_{\frac{3}{16}}(\xi)} \left(\int_{|\xi - z|}^{\frac{1}{4}} \frac{1}{r^2} G^r(z) \, dr \right) d\lambda(z) \\
&\geq C^{-1} \int_{B'_{\frac{3}{16}}(\xi)} \frac{d\lambda(z)}{|z - \xi|^{n-1}}.
\end{aligned}$$

For $\xi \in \overline{B}'_{\frac{1}{16}}$, this yields

$$\int_{B'_{\frac{1}{8}}} \frac{d\lambda(z)}{|z-\xi|^{n-1}} \leq \int_{B'_{\frac{3}{16}}(\xi)} \frac{d\lambda(z)}{|z-\xi|^{n-1}} \leq C.$$

We conclude that f is Lipschitz on $\overline{B}'_{\frac{1}{16}}$.

Because f is subharmonic, for $r \in (0, \frac{1}{4})$

$$\inf_{r \leq s \leq \frac{1}{4}} \frac{1}{s} \fint_{\partial B'_s} f \int_r^{\frac{1}{4}} \frac{ds}{s} \leq \int_0^{\frac{1}{4}} \frac{1}{r^2} \fint_{\partial B'_r} f$$

and

$$\left(\log \frac{1}{4r}\right) \inf_{r \leq s \leq \frac{1}{4}} \frac{1}{s} \fint_{\partial B'_s} f \leq C.$$

Let $\theta' > 0$ depend on θ to be chosen later. Let $r > 0$ be small enough so that $C(\log \frac{1}{4r})^{-1} \leq \frac{\theta'}{2}$. There exists $s \in [r, \frac{1}{4}]$ so that $\frac{1}{s} \fint_{\partial B'_s} f \leq \theta'$. Let g satisfy $\Delta g = 0$ in B'_s and $g = f$ on $\partial B'_s$, since f is Lipschitz in $\overline{B}'_{\frac{1}{16}}$ and $f(0) = 0$, $|f(x)| \leq C|x|$ for $x \in \overline{B}'_{\frac{1}{16}}$. Thus

$$g(0) = \fint_{\partial B'_s} |g| \leq C\theta' s,$$

and by the maximum principle

$$\sup_{B'_s} |g| \leq \sup_{\partial B'_s} |f| \leq Cs,$$

which implies

$$\sup_{B'_{\frac{s}{2}}} |\nabla g| \leq C \text{ and } \sup_{B'_{\frac{s}{2}}} |\nabla^2 g| \leq \frac{C}{s}.$$

Since f is subharmonic on B', this tells us that for $y \in B'_{\frac{s}{2}}$

$$f(y) \leq g(y) \leq C\theta' s + \langle \nabla g(0), y \rangle + \frac{C}{s}|y|^2.$$

That gives us

$$f(y) \leq \langle l, y \rangle + \frac{\theta}{2}\eta \text{ for } y \in B'_\eta,$$

where $l = \nabla g(0)$, $Q = 2C\sqrt{\theta'}$ and $\eta = \sqrt{\theta'}s$. □

Lemma 11. *Let $\Omega \subset \mathbb{R}^{n+1}$ be an unbounded domain and a set of locally finite perimeter such that $0 \in \partial\Omega$. Let v be a Green's function with pole at ∞ for Ω, let h be the associated Poisson kernel. Given $\theta \in (0,1)$ there exists $\sigma_\theta > 0$ and $\eta_\theta \in (0,1)$ so that if $\sigma \in (0, \sigma_\theta)$ and $\tau \in (0, \sigma_\theta \sigma^2)$ then for $Q_0 \in \partial\Omega$, $\rho > 0$ if $v \in F(\sigma;\sigma;\tau)$ in $B_\rho(Q_0)$ in the direction ν then $v \in F(\theta\sigma; 1; \tau)$ in the direction $B_{\eta\rho}(Q_0)$ in the direction $\overline{\nu}$ and $|\nu - \overline{\nu}| \leq C\sigma$.*

Proof: Assume that the lemma is false. Then there exists $\theta_0 \in (0,1)$ such that for every $\eta > 0$ (later we specify one) and every nonnegative decreasing sequence $\{\sigma_j\}$ there is a sequence $\{\tau_j\}$ with $\tau_j \sigma_j^{-2} \to 0$ so that

$$v \in F(\sigma_j; \sigma_j; \tau_j) \text{ in } B_{\rho_j}(Q_j) \text{ in the direction } \nu_j$$

but

$$v \notin F(\theta_0 \sigma_j; 1; \tau_j) \text{ in } B_{\eta\rho_j}(Q_j).$$

By Lemmas 8 and 10 we get a function f on $B'_{\frac{1}{16}}$ satisfying

$$f(y) \leq \langle l, y \rangle + \frac{\theta}{2}\eta \text{ for } y \in B'_\eta.$$

By Lemma 7, f is a uniform limit of the functions f_j^+. Therefore Lemma 10 yields that, for $\theta \in (0,1)$ there exists $\eta > 0$ so that for j large enough

$$f_j^+(y) \leq \langle l, y \rangle + \theta\eta \text{ for } y \in B'_\eta,$$

which by definition means that

$$v_j(X) = 0 \text{ for } X = (\overline{x}, x_{n+1}) \in B_\eta(0) \text{ with } x_{n+1} > \sigma_j \langle l, \overline{x} \rangle + \theta\eta\sigma_j.$$

Let $\overline{\nu} = (1 + \sigma_j^2 |l|^2)^{-\frac{1}{2}}(-\sigma_j l, 1)$ so the that previous line implies

$$v_j(X) = 0 \text{ for } X \in B_\eta(0) \text{ with } \langle X, \nu \rangle \geq \frac{\theta\eta\sigma_j}{(1 + \sigma_j^2 |l|^2)^{\frac{1}{2}}} \geq 2\theta\eta\sigma_j$$

for j large enough. Therefore $v \in F(2\theta\eta_j; 1; \tau_j)$ in $B_\eta(Q_0)$ in the direction ν. This contradicts the statement $v \notin F(\theta_0\sigma_j; 1; \tau_j)$ in $B_\eta(Q_0)$ in the case $\theta = \frac{\theta_0}{2}$. □

2.3 Main Theorem

Theorem 1. *Assume that $\Omega \subset \mathbb{R}^{n+1}$ is an unbounded NTA domain with locally finite perimeter satisfying the exterior separation property at large scales. Let v be a Green's function with pole at ∞, and let h be the associated Poisson kernel. Given $\delta > 0$ small enough (depending only on n) there exists $\epsilon = \epsilon(\delta) > 0$ such that if $\tau < \epsilon$ and*

$$h \geq 1 - \tau,$$

$$\limsup_{X \to Q \in \partial\Omega,\ X \in \Omega} |\nabla v(X)| \leq 1 + \tau$$

and

$$\liminf_{r \to \infty} \Theta(0, r) < \delta, \tag{2.19}$$

then

$$\Theta(Q, r) < 32\delta \ \textit{ for all } Q \in \partial\Omega \ \textit{ and for all } r > 0.$$

Proof: Fix $r > 0$. By Lemma 4, (2.19) implies that for $Q \in \partial\Omega$

$$\liminf_{r \to \infty} \Theta(Q, r) < 16\delta.$$

Let $\theta \in (0, \frac{1}{2})$ be such that

$$C\theta + 2\theta < 1, \tag{2.20}$$

where C is as in Proposition 6. Let $r > 0$ be given. Let σ_θ be as in Lemma (6), and set $\epsilon = \sigma_\theta$. Write $\sigma = 16\delta$, and assume that $\sigma < \sigma_\theta \sigma^2$ and $\tau < \epsilon$. By Proposition 1, $v \in F(\sigma; 1; \tau)$ in $B_\rho(Q)$ in the direction $\nu_0 = \nu_0(Q)$, for $Q \in \partial\Omega$ with $\rho \geq r$ sufficiently large. Next, Lemma 11 ensures that

$$v \in F(\theta\sigma; 1; \tau) \text{ in } B_{\eta\rho}(Q) \text{ in the direction } \nu_0.$$

Lemma 6 then guarantees that

$$v \in F(2\theta\sigma; C\theta\sigma, \tau) \text{ in } B_{\frac{\eta\rho}{2}}(Q) \text{ in the direction } \nu_1.$$

From this and the inequality (2.20) we conclude that

$$v \in F(\sigma;\sigma;\tau) \text{ in } B_{\frac{\eta\rho}{2}}(Q) \text{ in the direction } \nu_1.$$

Iterating the previous argument shows that, for all $k \in \mathbb{N}$ and for all $Q \in \partial\Omega$,

$$v \in F(\sigma;\sigma;\tau) \text{ in } B_{\left(\frac{\eta}{2}\right)^k\rho}(Q) \text{ in the direction } \nu_k,$$

for some $\nu_k \in \mathbb{S}^n$. Thus for $X \in B_{\left(\frac{\eta}{2}\right)^k\rho}(Q)$

$$v(X) = 0 \text{ for } \langle X - Q, \nu_k\rangle \geq \sigma\left(\frac{\eta}{2}\right)^k \rho \tag{2.21}$$

and

$$v(X) \geq -h(Q)\left[\langle X - Q, \nu_k\rangle + \sigma\left(\frac{\eta}{2}\right)^k \rho\right] \geq 0 \text{ for } \langle X - Q, \nu_k\rangle \leq -\sigma\left(\frac{\eta}{2}\right)^k \rho. \tag{2.22}$$

In particular if $L_k(Q)$ denotes the n-plane through Q orthogonal to ν_k, equations (2.21) and (2.22) imply

$$D\left[\partial\Omega \cap B_{\left(\frac{\eta}{2}\right)^k\rho}(Q); L_k(Q) \cap B_{\left(\frac{\eta}{2}\right)^k\rho}(Q)\right] \leq \sigma\left(\frac{\eta}{2}\right)^k \rho.$$

There is $k \geq 0$ so that $\left(\frac{\eta}{2}\right)^{k+1}\rho \leq r \leq \left(\frac{\eta}{2}\right)^k\rho$; define $r_k = \left(\frac{\eta}{2}\right)^k\rho$. For $P \in \partial\Omega \cap B_r(Q)$, there exists $\overline{Z} \in L_k(Q) \cap B_{r_k}(Q)$ so that $|Z - P| < \sigma r_k$. Note that

$$|Z - Q| \leq |Z - P| + |P - Q| < \sigma r_k + r.$$

Hence there exists Z' on the line segment from Q to Z such that $|Z' - Q| < r$ and $|Z' - Z| < \sigma r_k$; moreover, we see that

$$|Z' - P| \leq |Z - Z'| + |Z - P| < 2\sigma r_k.$$

Because Z and Q are both in the plane $L_k(Q)$ and the ball $B_r(Q)$, so is the point Z'. This proves

$$dist(P, L_k(Q) \cap B_r(Q)) < 2\sigma r_k \text{ for every } P \in \partial\Omega \cap B_r(Q). \tag{2.23}$$

Next, for $Z \in L_k(Q) \cap B_{r_k}(Q)$ there exists $Z' \in L_k(Q) \cap B_{r-\sigma r_k}(Q)$ so that $|Z - Z'| < \sigma r_k$. There exists $P \in \partial\Omega \cap B_{r_k}(Q)$ so that $|Z' - P| < \sigma r_k$. This give us $|Z - P| \leq |Z - Z'| + |Z' - P| < 2\sigma r_k$. Moreover,

$$|P - Q| \leq |P - Z'| + |Z' - Q| < \sigma r_k + (r - \sigma r_k) = r$$

Thus $P \in \partial\Omega \cap B_r(Q)$. Now we have proved

$$dist(Z, \partial\Omega \cap B_r(Q)) < 2\sigma r_k \text{ for every } Z \in L_k(Q) \cap B_r(Q). \tag{2.24}$$

Putting together inequalities (2.23) and (2.24) yields

$$\frac{1}{r} D\left[\partial\Omega \cap B_r(Q); L_{r_k}(Q) \cap B_r(Q)\right] \leq 2\sigma.$$

That is to say,

$$\Theta(Q, r) \leq 2\sigma \text{ for all } r > 0.$$

□

Chapter 3

A BOUNDED SETTING WITH A BMO CONDITION

In this chapter we prove estimates on the gradient of a Green's function that will allow us to show that a domain is "close to being a ball" in a geometric sense. We will make use of the following notation for $X \in \Omega \subset \mathbb{R}^{n+1}$:

$$\delta(X) = dist(X, \partial\Omega).$$

3.1 Crude Estimates

Lemma 12. *Let $G_0(X)$ be the Green's function for an NTA domain $\Omega \subset \mathbb{R}^{n+1}$ with pole at $0 \in \Omega$. Let ω^0 be the harmonic measure for Ω with pole at 0, and suppose it satisfies*

$$\omega^0(B_r(Q)) \leq Lr^n \quad \text{for all } r > 0 \text{ and } Q \in \partial\Omega.$$

Then for some $N = N(L) < \infty$ we have

$$|\nabla G_0| \leq N \quad \text{for } X \in \Omega \text{ with } \delta(X) \leq \frac{\delta(0)}{8}.$$

Proof: Let $R = \delta(0)$, so that $B_R(0) \subset \Omega$ and $\partial B_R(0) \cap \partial\Omega \neq \emptyset$. Let $r > 0$ and $Q \in \partial\Omega$ be such that $0 \notin B_r(Q)$, so that the Riesz decomposition theorem for subharmonic functions (see Theorem 6.18 in [7]) applied to G gives us

$$⨍_{\partial B_r(Q)} G(Z)\, d\mathcal{H}^n(Z) = \frac{1}{(n+1)\sigma_n} \int_0^r \frac{\omega^0(B_t(Q))}{t^n}\, dt.$$

Then because $\omega^0(B_t(Q)) \leq Lt^n$, we have

$$\fint_{\partial B_r(Q)} G(Z)\, d\mathcal{H}^n(Z) = \frac{Lr}{(n+1)\sigma_n}.$$

Let $X \in \Omega$ with $r = \delta(X) < \frac{R}{4}$. Select $Q \in \partial\Omega$ such that $|X - Q| = r$. Since $r < \frac{R}{4}$, we see that $0 \notin B_{4r}(Q)$, and the Riesz representation theorem for subharmonic functions implies

$$G(X) \leq \frac{(2r)^2 - |X-Q|^2}{2r(n+1)\omega_n} \int_{\partial B_{2r}(Q)} \frac{G(Z)}{|Z-X|^{n+1}}\, d\mathcal{H}^n(Z).$$

Since $|Z - X| \geq r$ for $X \in \partial B_r(Q)$, we now have

$$G(X) \leq \frac{3}{2} \fint_{\partial B_{2r}(Q)} G(Z)\, d\mathcal{H}^n(Z) \leq \frac{3L\delta(X)}{(n+1)\sigma_n}.$$

Thus, for $N = \frac{3L}{(n+1)\sigma_n}$, we have shown

$$G(X) \leq N\delta(X) \quad \text{for } X \in \Omega \text{ with } \delta(X) < \frac{R}{4}.$$

Standard estimates for harmonic functions on $\{Y \in \Omega; \delta(Y) < \frac{R}{8}\}$ now give

$$|\nabla G(X)| \leq N \quad \text{for } X \in \Omega \text{ with } \delta(X) < \frac{R}{8}.$$

□

Lemma 13. *Under the same hypothesis as Lemma 12, we see that*

$$\delta(0) \geq B = B(L).$$

Proof: Let $R = \delta(0)$. Let $\tilde{G}$ denote the Green's function for $B_R(0) \subset \Omega \subset \mathbb{R}^{n+1}$ with pole at 0. Let $F(X)$ denote the fundamental solution of the Laplacian:

$$\Phi(X) = \frac{C}{|X|^{n-1}}.$$

Here, $C = \frac{1}{(n-1)\sigma_n}$ is chosen so that $\Delta\Phi = -\delta_0$, the negative point mass at the origin, in the sense of distributions. Then let u denote the solution of

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = \Phi & \text{on } \partial\Omega \end{cases},$$

and let $\tilde{u}$ be the solution of

$$\begin{cases} \Delta \tilde{u} = 0 & \text{in } B_R(0) \\ \tilde{u} = \Phi & \text{on } \partial B_R(0) \end{cases}.$$

Then $G_0 = \Phi - u$ on Ω and $\tilde{G} = \Phi - \tilde{u}$ on $B_R(0)$. Therefore

$$G - \tilde{G} = (\Phi - u) - (\Phi - \tilde{u}) = \tilde{u} - u \qquad \text{on } B_R(0). \tag{3.1}$$

Because $B_R(0) \subset \Omega$ and Φ is radially decreasing, we see that

$$\max_{\partial\Omega} \Phi \leq \frac{C}{R^{n-1}}$$

because $\Phi(Q) \leq \frac{C}{R^{n-1}}$ for $Q \in \partial\Omega$. Therefore, by the maximum principle, we get

$$u \leq \frac{C}{R^{n-1}},$$

where the right side is the (constant) value of $\tilde{u}$ on $\partial B_R(0)$. Therefore $u \leq \tilde{u}$ on $\partial B_R(0)$. Inserting this into (3.1) yields

$$G \geq \tilde{G} \qquad \text{on } B_R(0).$$

Therefore, for any $Q \in \partial\Omega \cap \partial B_R(0)$ (which is nonempty), we get

$$\begin{aligned} |\nabla \tilde{G}(Q)| &= \lim_{h\to 0} \frac{\tilde{G}(\frac{R-h}{R}Q) - \tilde{G}(Q)}{h} \\ &= \lim_{h\to 0} \frac{\tilde{G}(\frac{R-h}{R}Q) - G(Q)}{h} \\ &\leq \limsup_{h\to 0} \frac{G(\frac{R-h}{R}Q) - G(Q)}{h} \\ &\leq \limsup_{h\to 0} \frac{1}{h} \int_{R-h}^{R} \left| \nabla G\left(\frac{R-t}{R}Q\right)\right| \, dt \\ &\leq N, \end{aligned}$$

with N as in Lemma 12. But we know from direct calculation that

$$|\nabla \tilde{G}(X)| = \frac{1}{\sigma_n |X|^n}.$$

Consequently,

$$|\nabla \tilde{G}(Q)| = \frac{1}{\sigma_n R^n}.$$

This gives us

$$\frac{1}{\sigma_n R^n} \leq N,$$

and therefore

$$R \geq (\sigma_n N)^{-\frac{1}{n}}.$$

□

The point of this argument was that Ω contains a ball, centered at the origin, with a radius bounded below in terms of the data L and the dimension of the space. Later in this chapter, we will improve this estimate by showing that Ω contains a ball whose radius is in fact much larger: close the radius of a ball with the same surface measure as $\partial\Omega$. However, these estimates will only work out if we add the following hypotheses regarding the domain: we will assume that Ω is non-tangentially accessible and Ahlfors regular.

3.2 A Finer Estimate on the Gradient of the Green's Function

The next technical lemma is an ingredient for the proof of Theorem 2.

Lemma 14. *Suppose that G_0 is a Green's function for a bounded domain $\Omega \subset \mathbb{R}^{n+1}$ that is NTA and Ahlfors regular, and k_0 is the corresponding Poisson kernel satisfying $k_0 \leq C < \infty$. Let $\vec{F}(Q)$ be the nontangential limit function of ∇G_0 on $\partial\Omega$. Then since $k_0 \in L^2_{loc}(d\mathcal{H}^n)$, for $\mathcal{H}^n$ a.e. $Q \in \partial\Omega$ we have*

$$\vec{F}(Q) = k_0(Q)\vec{n}(Q)$$

where $\vec{n}$ is the inward-unit normal vector to $\partial\Omega$.

The proof of this lemma is contained in [14]. It is Lemma 3.2 in that article.

Theorem 2. *Let $\Omega \subset \mathbb{R}^{n+1}$ be an Ahlfors regular, non-tangentially accessible domain with NTA constant M, and assume $0 \in \Omega$. Let $G_0(X)$ denote the Green's function for Ω with pole at 0, let ω^0 denote the harmonic measure for Ω with pole at the origin, and let k_0 denote the associated Poisson kernel. Suppose that*

$$\omega^0(B_r(Q)) \leq Lr^n \quad \textit{for all } r > 0 \textit{ and } Q \in \partial\Omega.$$

Then there exists a constant $\alpha > 0$ depending only on M such that for $X \in \Omega$ with $\delta(X) < \frac{\delta(0)}{16}$ we have

$$|\nabla G_0(X)| \leq \int_{\partial\Omega} k_0(Q) d\omega^X + C\delta(X)^\alpha. \tag{3.2}$$

The constant C depends only on n, M and L.

Proof: Fix $X \in \Omega$ with $\delta(X) < \frac{\delta(0)}{16}$, and select $Q_0 \in \partial\Omega$ with $|Q_0 - X| = \delta(X)$. Let $\phi \in C_c^\infty\left(B_{\frac{\delta(0)}{4}}(Q_0)\right)$, with

$$\phi(Z) = 1 \quad \text{for} \quad |Z - Q_0| < \frac{\delta(0)}{8}, \tag{3.3}$$

$$0 \leq \phi \leq 1, \tag{3.4}$$

and

$$|\nabla\phi| \leq \frac{C}{\delta(0)} \quad \text{and} \quad |\Delta\phi| \leq \frac{C}{\delta(0)^2}. \tag{3.5}$$

In particular, $\phi = 0$ in $B_{\delta(0)/4}(0)$.

Let G_Z denote the Green's function for Ω with pole at Z. For $Z \in \Omega$ define $\omega_0 : \Omega \to \mathbb{R}^{n+1}$ by

$$\omega_0(Z) = \int_\Omega G_Z(Y)\Delta[\phi(Y)\nabla G_0(Y)]\, dY.$$

Here, $\Delta[\phi(Y)\nabla G_0(Y)]$ denotes the vector whose i^{th} entry is $\Delta[\phi(Y)\frac{\partial G_0}{\partial Y_i}(Y)]$. Hence for $Y \neq 0$ we have

$$\begin{aligned}(\Delta[\phi(Y)\nabla G_0(Y)])_i &= \Delta[\phi(Y)\frac{\partial G_0}{\partial Y_i}(Y)] \\ &= (\Delta\phi(Y))\frac{\partial G_0}{\partial Y_i}(Y) + 2\nabla\phi(Y)\cdot\nabla\frac{\partial G_0}{\partial Y_i}(Y) + \phi(Y)\Delta\frac{\partial G_0}{\partial Y_i}(Y) \\ &= (\Delta\phi(Y))\frac{\partial G_0}{\partial Y_i}(Y) + 2\nabla\phi(Y)\cdot\nabla\frac{\partial G_0}{\partial Y_i}(Y).\end{aligned}$$

In the last line we used the observation that $\frac{\partial G_0}{\partial Y_i}(Y)$ is harmonic away from the origin. For the sake of convenience, we will use the following notation: Let $\nabla\phi(Y)\cdot\nabla(\nabla G_0(Y))$ denote the vector whose i^{th} entry is $\nabla\phi(Y)\cdot\nabla\frac{\partial G_0}{\partial Y_i}(Y)$. Then we have

$$\Delta[\phi(Y)\nabla G_0(Y)] = (\Delta\phi(Y))\nabla G_0 + 2\nabla\phi(Y)\cdot\nabla[\nabla G_0(Y)].$$

Now we define

$$\omega_0^1(Z) = \int_\Omega G_Z(Y)\Delta\phi(Y)\nabla G_0(Y)\, dY$$

and

$$\omega_0^2(Z) = 2\int_\Omega G_Z(Y)\nabla\phi(Y)\cdot\nabla(\nabla G_0(Y))\, dY.$$

Thus we obtain $\omega_0 = \omega_0^1 + \omega_0^2$. These vector-valued integrals converge for all $Z \in \Omega$ because ϕ is supported away from the singularity of G_0 at the origin, so each integrand is a Green's function $G_Z(Y)$ multiplied by a smooth, compactly supported function. We will be able to analyze each of the terms ω_0^1 and ω_0^2 separately. We will demonstrate that these terms have the kind of decay at $\partial\Omega$ that we need in the last term in (3.2).

Standard estimates for harmonic functions give us

$$|\nabla G_0(Y)| \leq C\frac{G_0(Y)}{\delta(Y)} \quad \text{and} \quad |\nabla^2 G_0(Y)| \leq C\frac{G_0(Y)}{\delta(Y)^2}, \tag{3.6}$$

with $C = C(n)$. The second inequality implies in particular that

$$|\nabla(\nabla G_0(Y))| \leq C\frac{G_0(Y)}{\delta(Y)^2}.$$

If we let $R = \frac{\delta(0)}{8}$, we have $\text{spt}\nabla\phi, \text{spt}\Delta\phi \subset B_{2R}(Q_0) \setminus \overline{B_R(Q_0)}$, because ϕ is constant outside $B_{2R}(Q_0)$ and inside $B_R(Q_0)$. Consequently, and because of (3.5),

$$\begin{aligned} |\omega_0^2(X)| &\leq 2\int_\Omega |G_X(Y)||\nabla\phi(Y)||\nabla(\nabla G_0(Y)|\, dY \\ &\leq \frac{C}{\delta(0)}\int_{R\leq|Y-Q_0|\leq 2R} G_X(Y)\frac{G_0(Y)}{\delta^2(Y)}\, dY, \end{aligned} \tag{3.7}$$

with $C = C(n)$. Next, let $A_s = A(Q_0, \frac{s}{2})$ be an interior NTA point for Q_0, so that

$$\frac{s}{M} \leq |A_s - Q_0| \leq s$$

and

$$\delta(A_s) \geq \frac{s}{M}.$$

Then for $Y \in \Omega \cap B(Q_0, 2R) \setminus \overline{B(Q_0, R)}$, we have

$$G_X(Y) \leq C\left[\frac{\delta(X)}{\delta(0)}\right]^\alpha G_{A_R}(Y) \tag{3.8}$$

with $C = C(M)$, for some $0 < \alpha < 1$ (see Lemma 4.1 in [9]).

From the comparison principle (Lemma 4.10 in [9]), we see for the same Y that

$$\frac{G_{A_R}(Y)}{G_{A_R}(A_{2R})} \leq C\frac{G_0(Y)}{G_0(A_{2R})}, \qquad C = C(M).$$

Given this, we have

$$G_{A_R}(Y) \leq CG_{A_R}(A_{2R})\frac{G_0(Y)}{G_0(A_2R)} \leq \frac{C}{\delta(0)^{n-1}}\frac{G_0(Y)}{G_0(A_2R)}, \qquad C = C(M). \tag{3.9}$$

Consequently,

$$
\begin{aligned}
|\omega_0^2(X)| &\leq \frac{C}{\delta(0)} \int_{R\leq|Y-Q_0|\leq 2R} G_X(Y)\frac{G_0(Y)}{\delta^2(Y)}\,dY \\
&\leq \frac{C}{\delta(0)} \int_{R\leq|Y-Q_0|\leq 2R} \left[\frac{\delta(X)}{\delta(0)}\right]^{\alpha} G_{A_R}(Y)\frac{G_0(Y)}{\delta^2(Y)}\,dY \\
&\leq \frac{C}{\delta(0)} \left[\frac{\delta(X)}{\delta(0)}\right]^{\alpha} \int_{R\leq|Y-Q_0|\leq 2R} \frac{C}{\delta(0)^{n-1}}\frac{G_0(Y)}{G(A_{2R})}\frac{G_0(Y)}{\delta^2(Y)}\,dY \\
&= \frac{C}{\delta(0)^n} \left[\frac{\delta(X)}{\delta(0)}\right]^{\alpha} \frac{1}{G(A_{2R})} \int_{R\leq|Y-Q_0|\leq 2R} \frac{G_0^2(Y)}{\delta^2(Y)}\,dY. \qquad (3.10)
\end{aligned}
$$

We used (3.7) to obtain the first line above, (3.8) to obtain the second line, and (3.9) to obtain the third line.

Next, we would like to estimate $\int_{R\leq|Y-Q_0|\leq 2R} \frac{G^2(Y)}{\delta^2(Y)}\,dY$. Since Ω is NTA, there exists $C = C(M) > 1$ such that

$$
C^{-1} < \frac{\omega^0(B_{\delta(Y)}(Q))}{\delta(Y)^{n-1}G_Y(0)} < C,
$$

according to Lemma 4.8 in [9]. Thus

$$
G_Y(0) < C\frac{\omega^0(B_{\delta(Y)}(Q))}{\delta(Y)^{n-1}},
$$

hence

$$
\frac{G_Y(0)}{\delta(Y)} \leq C\frac{\omega^0(B_{\delta(Y)}(Q))}{\delta(Y)^n}.
$$

Because of the symmetry of the Green's function,

$$
\frac{G_0(Y)}{\delta(Y)} \leq C\frac{\omega^0(B_{\delta(Y)}(Q))}{\delta(Y)^n} \leq CL. \qquad (3.11)
$$

The second inequality in the last line makes use of the hypothesis that for all $r > 0$ and $Q \in \partial\Omega$, we have $\omega^0(B(Q,r))$. This now gives us

$$
\int_{R\leq|Y-Q_0|\leq 2R} \frac{G_0^2(Y)}{\delta^2(Y)}\,dY \leq CL^2\delta(0)^{n+1} \qquad (3.12)
$$

(having recalled that $R = \frac{\delta(0)}{8}$), and this implies

$$
\begin{aligned}
|\omega_0^2(X)| &\leq \frac{C}{\delta(0)^n}\left[\frac{\delta(X)}{\delta(0)}\right]^\alpha \frac{1}{G(A_{2R})}\int_{R\leq|Y-Q_0|\leq 2R}\frac{G_0^2(Y)}{\delta^2(Y)}\,dY \qquad \text{(by (3.10))}\\
&\leq \frac{CL^2}{\delta(0)^n}\left[\frac{\delta(X)}{\delta(0)}\right]^\alpha \frac{1}{G_0(A_{2R})}\delta(0)^{n+1} \qquad \text{(by (3.12))}\\
&= \frac{CL^2\delta(0)}{G_0(A_{2R})}\left[\frac{\delta(X)}{\delta(0)}\right]^\alpha. \qquad (3.13)
\end{aligned}
$$

The point is that this term will vanish like $\delta(X)^\alpha$ as $\delta(X) \to 0$.

Next we try to estimate

$$\omega_0^1(X) = \int_\Omega G_X(Y)\Delta\phi(Y)\nabla G_0(Y)\,dY.$$

Using the fact that spt$\Delta\phi \subset \{R \leq |Y - Q_0| \leq 2R\}$, we get

$$
\begin{aligned}
|\omega_0^1(X)] &= \int_{R\leq|Y-Q_0|\leq 2R} G_X(Y)|\Delta\phi(Y)||\nabla G_0(Y)|\,dY\\
&\leq \frac{C}{\delta(0)^2}\int_{R\leq|Y-Q_0|\leq 2R} G_X(Y)|\nabla G_0(Y)|\,dY \qquad \text{(by (3.5))}\\
&\leq \frac{C}{\delta(0)^2}\int_{R\leq|Y-Q_0|\leq 2R} G_X(Y)\frac{G_0(Y)}{\delta(Y)}\,dY \qquad \text{(by (3.6))}\\
&\leq \frac{C}{\delta(0)^2}\left[\frac{\delta(X)}{\delta(0)}\right]^\alpha\int_{R\leq|Y-Q_0|\leq 2R} G_{A_R}(Y)\frac{G_0(Y)}{\delta(Y)}\,dY \qquad \text{(by (3.8))}\\
&\leq C\left[\frac{\delta(X)}{\delta(0)}\right]^\alpha \frac{1}{\delta(0)^{n+1}}\frac{1}{G_0(A_{2R})}\int_{R\leq|Y-Q_0|\leq 2R}\frac{G_0(Y)^2}{\delta(Y)}\,dY \qquad \text{(by (3.9))}\\
&\leq CL\left[\frac{\delta(X)}{\delta(0)}\right]^\alpha \frac{1}{\delta(0)^{n+1}}\frac{1}{G_0(A_{2R})}\int_{R\leq|Y-Q_0|\leq 2R} G_0(Y)\,dY \qquad \text{(by (3.11))}\\
&\leq CL^2\left[\frac{\delta(X)}{\delta(0)}\right]^\alpha \frac{1}{\delta(0)^{n+1}}\frac{1}{G_0(A_{2R})}\int_{R\leq|Y-Q_0|\leq 2R} \delta(Y)\,dY \qquad \text{(by (3.11) again)}\\
&\leq CL^2\left[\frac{\delta(X)}{\delta(0)}\right]^\alpha \frac{1}{\delta(0)^{n+1}}\frac{1}{G_0(A_{2R})}\int_{R\leq|Y-Q_0|\leq 2R} \frac{\delta(0)}{4}\,dY\\
&\qquad\qquad \text{(since } \delta(Y) \leq |Y-Q_0| \leq \tfrac{\delta(0)}{4}\text{)}\\
&\leq CL^2\left[\frac{\delta(X)}{\delta(0)}\right]^\alpha \frac{1}{\delta(0)^{n+1}}\frac{1}{G_0(A_{2R})}\delta(0)^{n+2}\\
&= CL^2\left[\frac{\delta(X)}{\delta(0)}\right]^\alpha \frac{\delta(0)}{G_0(A_{2R})}.
\end{aligned}
$$

The value of the constant C changed from line to line in this calculation, but at each step it depended only on n and on the NTA constant M. Thus we have

$$|\omega_0^1(X)| \le CL^2 \frac{\delta(0)}{G_0(A_{2R})} \left[\frac{\delta(X)}{\delta(0)}\right]^\alpha, \qquad C = C(n, M).$$

Combining this with (3.13) gives us

$$|\omega_0(X)| \le CL^2 \frac{\delta(0)}{G_0(A_{2R})} \left[\frac{\delta(X)}{\delta(0)}\right]^\alpha, \qquad C = C(n, M).$$

Next, let

$$f(X) = \phi(X)\nabla G_0(X) + \omega_0(X). \tag{3.14}$$

Note that $f(X)$ is a harmonic vector-valued function in Ω since

$$\begin{aligned}\Delta f(X) &= \Delta(\phi(X)\nabla G(X)) + \Delta\omega_0(X)\\ &= \Delta(\phi(X)\nabla G(X)) + \Delta \int_\Omega G_X(Y)\Delta[\phi(Y)\nabla G(Y)dY\\ &= \Delta(\phi(X)\nabla G(X)) - \Delta[\phi(X)\nabla G(X)]\\ &= 0.\end{aligned}$$

In the third line we used the fact that $\Delta G_X(Y) = -\delta_X$, the point mass at X. Observe also that

$$f(X) = 0 \quad \text{on } \partial\Omega \setminus B_{\delta(0)/4}(Q_0)$$

because spt $\phi \subset B_{\delta(0)/4}(Q_0)$ and spt $\omega_0 \subset B_{\delta(0)/4}(Q_0)$.

According to Lemma 14, $N(\phi\nabla G_0) \in L^1(\partial\Omega; \omega^Z)$ for all $Z \in \Omega$. (Here, $N(g)$ denotes the non-tangential limit function on $\partial\Omega$ arising from a function g on Ω.) Then because ω_0 is bounded, we also have $N(\omega_0) \in L^1(\partial\Omega; \omega^Z)$ for all $Z \in \Omega$. Thus $N(f(X)) \in L^1(\partial\Omega; \omega^Z)$ for all $Z \in \Omega$. Consequently,

$$f(X) = \int_{\partial\Omega} F(Q)\, d\omega^X \qquad \text{for all } X \in \Omega,$$

where $F(Q)$ is the non-tangential limit function for $f(X)$, which again by Lemma 14 satisfies

$$|F(Q)| = \phi(Q)k_0(Q).$$

So for $X \in B_{\delta(0)/4}(Q_0)$, we know that $\phi(X) = 1$, giving us

$$\begin{aligned}
|\nabla G(X)| &= |f(X) - \omega_0(X)| \\
&\leq |f(X)| + |\omega_0(X)| \\
&= \left| \int_{\partial\Omega} F(Q)\, d\omega^X \right| + |\omega_0(X)| \\
&\leq \int_{\partial\Omega} \phi(Q) k_0(Q)\, d\omega^X + |\omega_0(X)| \\
&\leq \int_{\partial\Omega} k_0(Q)\, d\omega^X + |\omega_0(X)| \quad \text{(since } 0 \leq \phi \leq 1\text{)} \\
&\leq \int_{\partial\Omega} k_0(Q)\, d\omega^X + \frac{CL^2}{\delta(0)} G_0(A_{2R}) \left[\frac{\delta(X)}{\delta(0)}\right]^\alpha . \qquad (3.15)
\end{aligned}$$

Note that

$$\begin{aligned}
G_0(A_{2R}) &\leq \sup_{\{\delta(Y) < 2R\}} |\nabla G_0(Y)| \delta(A_{2R}) \\
&\leq 2NR \quad \text{(by Lemma 12)} \\
&= \frac{N\delta(0)}{4}.
\end{aligned}$$

Inserting this into (3.15) gives us

$$|\nabla G_0(X)| \leq \int_{\partial\Omega} k_0(Q)\, d\omega^X + CNL^2 \left[\frac{\delta(X)}{\delta(0)}\right]^\alpha \leq \int_{\partial\Omega} k_0(Q)\, d\omega^X + \frac{CNL^2}{B^\alpha} \delta(X)^\alpha,$$

using Lemma 13 to obtain the final inequality. Because C and α depend only on the NTA constant M, this is exactly what we wanted to prove.

□

3.3 The Inside Ball

In the remainder of this chapter, we prove that if the logarithm of the Poisson kernel has small BMO-seminorm, relative to other geometric constants for $\partial\Omega$, then Ω is close to being a ball. In particular, we prove that Ω contains a ball, and that it is contained in a ball, and that the radii of these two balls are very close to one another. Ahlfors regularity plays in important role here.

The following result makes use of the gradient estimates above on a Green's function near the boundary to show that Ω must contain a relatively large ball. The radius of the ball must be, in fact, close to the radius of the ball whose surface area is the same as that of $\partial\Omega$. We define the function space $BMO(\partial\Omega)$ as follows: $f \in BMO(\partial\Omega)$ if and only if

$$\| f \|_{BMO} := \sup_{Q\in\partial\Omega} \sup_{r>0} ⨍ \left| f - ⨍ f \, d\mathcal{H}^n \right| \, d\mathcal{H}^n < \infty.$$

This quantity is a seminorm: $\| f \|_{BMO} = 0$ if and only if $f = k$ $\mathcal{H}^n$-almost everywhere for some constant k. Because under our hypotheses, $(\partial\Omega, \mathcal{H}^n)$ is a homogeneous space, the John-Nirenberg Inequality holds:

$$\mathcal{H}^n \left(\left| f - ⨍_{\partial\Omega} f d\mathcal{H}^n \right| > \epsilon \right) \leq \Gamma e^{\frac{-\gamma\epsilon}{\|f\|_{BMO}}} \mathcal{H}^n(\partial\Omega) \quad \text{for} \quad f \in BMO(\partial\Omega). \tag{3.16}$$

Here, $\Gamma > 0$ and $\gamma > 0$ depend only on n, $\mathcal{H}^n(\partial\Omega)$ and the Ahlfors regularity constant A.

Theorem 3. *Consider a bounded, non-tangentially accessible domain $\Omega \subset \mathbb{R}^{n+1}$ with NTA constant $M \geq 1$ such that $0 \in \Omega$. Let ω denote the harmonic measure for Ω with pole at 0, and let h be the corresponding Poisson kernel. Write $R = \delta(0)$, and assume that*

(i) $\partial\Omega$ *is bounded and Ahlfors regular,*

i.e. $\dfrac{1}{A} \leq \dfrac{\mathcal{H}^n(B(Q,r)\cap\partial\Omega)}{r^n} \leq A,$ *for some* $A \geq 1$

and for all $Q \in \partial\Omega,\ 0 < r < diam(\Omega);$

(ii) $\log h \in BMO(\partial\Omega, \mathcal{H}^n),$ *with* $\| \log h \|_{BMO} = \epsilon_0^2$ *small*; *and*

(iii) $\omega(B(Q,r)) \leq Lr^n$ *for all* $Q \in \partial\Omega, r \geq 0.$

There exists a positive function F defined on $\mathbb{R}^+$ with $\lim_{\eta\searrow 0} F(\eta) = 0$ such that

$$R \geq \left[\sigma_n \left(\frac{e^{\epsilon_0}}{\mathcal{H}^n(\partial\Omega)} + F(\epsilon_0) \right) \right]^{-\frac{1}{n}}.$$

The function F depends only on the geometric data n, M, A, L and $\mathcal{H}^n(\partial\Omega)$.

Proof:

First observe that the assumption $\omega(B(Q,r)) \leq Lr^n$ implies $h \leq LA$ for $\mathcal{H}^n$-a.e. $Q \in \partial\Omega$, since for all Lebesgue points Q of h we have

$$\frac{1}{\mathcal{H}^n(B_r(Q))} \int_{B_r(Q)} h \, d\mathcal{H}^n \to h(Q) \quad \text{as } r \to 0$$

and

$$\frac{1}{\mathcal{H}^n(B_r(Q))} \int_{B_r(Q)} h \, d\mathcal{H}^n = \frac{\omega(B_r(Q))}{\mathcal{H}^n(B_r(Q))} \leq \frac{Lr^n}{\frac{1}{A} r^n} = LA.$$

Let G denote the Green's function for Ω with pole at 0.

Consider the domain $\Omega_t = \{X \in \Omega; G(X) > t\} \cup \{0\}$. Let G^t denote the Green's function for Ω_t with pole at 0. Observe that $G^t(X) = \max\{G(X) - t, 0\}$. The boundary of Ω_t is $\{X \in \Omega; G(X) = t\}$. According to Theorem 2, we have

$$\begin{aligned}
|\nabla G(X)| &\leq \int_{\partial\Omega} h(Q) \, d\omega^X + C\delta(X)^\alpha \\
&= \int_{\{h \leq \frac{e^{\epsilon_0}}{\mathcal{H}^n(\partial\Omega)}\}} h(Q) \, d\omega^X + \int_{\{h > \frac{e^{\epsilon_0}}{\mathcal{H}^n(\partial\Omega)}\}} h(Q) \, d\omega^X + C\delta(X)^\alpha \\
&\leq \int_{\{h \leq \frac{e^{\epsilon_0}}{\mathcal{H}^n(\partial\Omega)}\}} \frac{e^{\epsilon_0}}{\mathcal{H}^n(\partial\Omega)} \, d\omega^X + \int_{\{h > \frac{e^{\epsilon_0}}{\mathcal{H}^n(\partial\Omega)}\}} h(Q) \, d\omega^X + C\delta(X)^\alpha \\
&\leq \int_{\partial\Omega} \frac{e^{\epsilon_0}}{\mathcal{H}^n(\partial\Omega)} \, d\omega^X + \int_{\{h > \frac{e^{\epsilon_0}}{\mathcal{H}^n(\partial\Omega)}\}} h(Q) \, d\omega^X + C\delta(X)^\alpha \\
&= \frac{e^{\epsilon_0}}{\mathcal{H}^n(\partial\Omega)} + \int_{\{h > \frac{e^{\epsilon_0}}{\mathcal{H}^n(\partial\Omega)}\}} h(Q) \, d\omega^X + C\delta(X)^\alpha \quad \text{(since } \omega^X(\partial\Omega) = 1\text{)} \\
&\leq \frac{e^{\epsilon_0}}{\mathcal{H}^n(\partial\Omega)} + LA \int_{\{h > \frac{e^{\epsilon_0}}{\mathcal{H}^n(\partial\Omega)}\}} d\omega^X + C\delta(X)^\alpha \quad \text{(since } h \leq LA\text{)} \\
&= \frac{e^{\epsilon_0}}{\mathcal{H}^n(\partial\Omega)} + LA\omega^X \left(\left\{ h > \frac{e^{\epsilon_0}}{\mathcal{H}^n(\partial\Omega)} \right\} \right) + C(n, M, L)\delta(X)^\alpha. \qquad (3.17)
\end{aligned}$$

We would like to estimate the size of $\omega^X(\{h > \frac{e^{\epsilon_0}}{\mathcal{H}^n(\partial\Omega)}\})$ in order to continue.

Suppose that for some $Q \in \partial\Omega$ we have

$$\log h(Q) - ⨍_{\partial\Omega} \log h \, d\mathcal{H}^n \leq \epsilon_0.$$

Then

$$\log h(Q) \leq \fint_{\partial\Omega} \log h \, d\mathcal{H}^n + \epsilon_0,$$

and exponentiating both sides yields

$$h(Q) \leq e^{\epsilon_0} \exp\left(\fint_{\partial\Omega} \log h \, d\mathcal{H}^n \right) \leq e^{\epsilon_0} \fint_{\partial\Omega} h \, d\mathcal{H}^n,$$

with the last estimate following from Jensen's Inequality. Thus we have $h(Q) \leq \frac{e^{\epsilon_0}}{\mathcal{H}^n(\partial\Omega)}$. This argument proves that

$$\begin{aligned}\left\{ h > \frac{e^{\epsilon_0}}{\mathcal{H}^n(\partial\Omega)} \right\} &\subset \left\{ \log h(Q) - \fint_{\partial\Omega} \log h \, d\mathcal{H}^n > \epsilon_0 \right\} \\ &\subset \left\{ \left| \log h(Q) - \fint_{\partial\Omega} \log h \, d\mathcal{H}^n \right| > \epsilon_0 \right\}.\end{aligned}$$

We can now employ the John-Nirenberg Inequality for BMO functions to estimate

$$\mathcal{H}^n \left(\left\{ \left| \log h(Q) - \fint_{\partial\Omega} \log h \, d\mathcal{H}^n \right| > \epsilon_0 \right\} \right),$$

and then using the hypothesis that $\omega(B_r(Q)) \leq Lr^n$, we will be able to turn this into an estimate for

$$\omega \left(\left\{ \left| \log h(Q) - \fint_{\partial\Omega} \log h \, d\mathcal{H}^n \right| > \epsilon_0 \right\} \right).$$

To make use of this fact about ω, we need to be more specific about the point X so that we can employ Harnack's inequality to derive a similar estimate for ω^X.

Let $G = \Phi - \phi$ on Ω, where Φ is the fundamental solution of Laplace's equation on $\mathbb{R}^{n+1}$, $\Phi(Y) = \frac{C}{|Y|^{n-1}}$, and ϕ is the correction function satisfying

$$\begin{cases} \Delta\phi = 0 & \text{on } \Omega \\ \phi = \Phi & \text{on } \partial\Omega \end{cases}$$

Notice that Φ is positive, and therefore so is ϕ by the strong maximum principle. Thus we have for $Y \in \partial B_{\frac{\delta(0)}{4}}(0)$ that

$$G(Y) = \Phi(Y) - \phi(Y) \leq \Phi(Y) = \frac{C}{\left(\frac{\delta(0)}{4}\right)^{n-1}} = \frac{4^{n-1}C}{\delta(0)^{n-1}}.$$

From Lemma 13 we have

$$\delta(0) \geq \left[\frac{\sigma_n N}{(n-1)}\right]^{\frac{1}{n}}.$$

Consequently,

$$G(Y) \leq \frac{4^{n-1}\sigma_n^{\frac{1-2n}{n}}(n-1)^{\frac{n-1}{n}}}{N^{\frac{n-1}{n}}} = C(n,L) \quad \text{for } Y \in \partial B_{\frac{\delta(0)}{4}}(0).$$

With this estimate, and the fact that $G = 0$ on $\partial\Omega$, the maximum principle also gives us

$$G(Y) \leq C(n,L) \quad \text{on } \mathbb{R}^{n+1} \setminus B_{\frac{\delta(0)}{4}}(0),$$

where we have extended G to be zero on Ω^c.

If we assume that t is small enough so that $\partial\Omega_t \subset \mathbb{R}^{n+1} \setminus B_{\frac{\delta(0)}{4}}(0)$, Lemma (4.1) in [9] now gives us

$$G(X) \leq MC(n,L)\delta(X)^{\beta} \quad \text{for } X \in \partial\Omega_t,$$

where $\beta > 0$ depends only on M. That is to say,

$$t \leq MC(n,L)\delta(X)^{\beta},$$

so

$$\delta(X) \geq \left(\frac{t}{MC(n,L)}\right)^{\frac{1}{\beta}}.$$

Note also that for small t,

$$\delta(0) \geq \left(\frac{t}{MC(n,L)}\right)^{\frac{1}{\beta}},$$

so

$$\min(\delta(X), \delta(0)) \geq \left(\frac{t}{MC(n,L)}\right)^{\frac{1}{\beta}}.$$

Let

$$R_t = \sup\{r > 0; B_r(0) \subset \Omega_t\}.$$

Then $\partial\Omega_t \cap \partial B_{R_t}(0) \neq \emptyset$. Fix $X \in \partial\Omega_t \cap \partial B_{R_t}(0)$.

The distance from 0 to X is

$$|X-0| = R_t = \frac{R_t}{\left(\frac{t}{MC(n,L)}\right)^{\frac{1}{\beta}}}\left(\frac{t}{MC(n,L)}\right)^{\frac{1}{\beta}}.$$

Because Ω is an NTA domain, there is a Harnack chain from 0 to X of length at most Mk, where k is the least integer greater than $\log_2\left(\frac{R_t}{\left(\frac{t}{MC(n,L)}\right)^{\frac{1}{\beta}}}\right)$. In particular, if t is small enough, depending on $M,\ L$ and n, then

$$k \leq 2\log_2\left(\frac{R_t}{\left(\frac{t}{MC(n,L)}\right)^{\frac{1}{\beta}}}\right).$$

Now we apply Harnack's inequality to each ball in the Harnack chain to obtain that, for any nonnegative harmonic function u on Ω,

$$u(X) \leq M^k u(0) = 2^{(\log_2 M)k} u(0) \leq \left(\frac{R_t}{\left(\frac{t}{MC(n,L)}\right)^{\frac{1}{\beta}}}\right)^{2\log_2 M} u(0).$$

Since $\omega^Y\left(\left\{\left|\log h - \fint_{\partial\Omega}\log h\, d\mathcal{H}^n\right| > \epsilon_0\right\}\right)$ is a nonnegative harmonic function of Y, it follows that

$$\begin{aligned}\omega^X&\left(\left\{\left|\log h - \fint_{\partial\Omega}\log h\, d\mathcal{H}^n\right| > \epsilon_0\right\}\right)\\ &\leq \left(\frac{R_t}{\left(\frac{t}{MC(n,L)}\right)^{\frac{1}{\beta}}}\right)^{2\log_2 M} \omega(\{|\log \mathcal{H}^n(\partial\Omega)h| \geq \epsilon_0\}).\end{aligned}$$

We can apply (3.16) to get

$$\begin{aligned}\omega\left(\left\{\left|\log h - \fint_{\partial\Omega}\log h\, d\mathcal{H}^n\right| > \epsilon_0\right\}\right) &\leq LA\mathcal{H}^n\left(\left\{\left|\log h - \fint_{\partial\Omega}\log h\, d\mathcal{H}^n\right| > \epsilon_0\right\}\right)\\ &\leq \Gamma e^{\frac{-\gamma\epsilon_0}{\|\log h\|_*}}\mathcal{H}^n(\partial\Omega).\end{aligned}$$

Therefore

$$\omega^X(\{|\log \mathcal{H}^n(\partial\Omega)h| > \epsilon_0\}) \leq \left(\frac{R_t}{\left(\frac{t}{MC(n,L)}\right)^{\frac{1}{\beta}}}\right)^{2\log_2 M} \Gamma e^{\frac{-\gamma\epsilon_0}{\|\log h\|_*}} \mathcal{H}^n(\partial\Omega)$$
$$= C(n,M,L) R_t^{2\log_2 M} t^{-\frac{2\log_2 M}{\beta}} \Gamma e^{\frac{-\gamma\epsilon_0}{\|\log h\|_*}} \mathcal{H}^n(\partial\Omega).$$

We now take $t = \epsilon_0 = \sqrt{\| \log h \|_*}$ to get

$$\omega^X(\{|\log \mathcal{H}^n(\partial\Omega)h| > \epsilon_0\}) \leq C(n,M,L)\Gamma R_t^{2\log_2 M} \epsilon_0^{-\frac{2\log_2 M}{\beta}} e^{\frac{-\gamma}{\epsilon_0}} \mathcal{H}^n(\partial\Omega).$$

The isoperimetric inequality gives us an *a priori* bound on R_t:

$$R_t \leq \left(\frac{\mathcal{H}^n(\partial\Omega)}{\sigma_n}\right)^{\frac{1}{n}}.$$

Thus we have

$$\omega^X(\{|\log \mathcal{H}^n(\partial\Omega)h| > \epsilon_0\}) \leq C(n,M,L,\mathcal{H}^n(\partial\Omega))\Gamma \epsilon_0^{-\frac{2\log_2 M}{\beta}} e^{\frac{-\gamma}{\epsilon_0}} \mathcal{H}^n(\partial\Omega).$$

We also know that the constants Γ and γ from the John-Nirenberg inequality depend only on n, $\mathcal{H}^n(\partial\Omega)$ and A (the Ahlfors-regularity constant), so we obtain

$$\omega^X(\{|\log \mathcal{H}^n(\partial\Omega)h| > \epsilon_0\}) \leq C(n,M,L,A,\mathcal{H}^n(\partial\Omega)) \left(\epsilon_0^{-\frac{2\log_2 M}{\beta}} e^{\frac{-\gamma}{\epsilon_0}}\right).$$

Write

$$F_1(\epsilon) = LAC(n,M,L,A,\mathcal{H}^n(\partial\Omega)) \left(\epsilon^{-\frac{2\log_2 M}{\beta}} e^{\frac{-\gamma}{\epsilon}}\right),$$

where $C(n,M,L,A,\mathcal{H}^n(\partial\Omega))$ is as in the previous line. This is a power function of ϵ multiplied by a function that decays exponentially as $\epsilon \to 0$, so the whole quantity approaches zero as ϵ does. We can now write

$$\omega^X(\{|\log \mathcal{H}^n(\partial\Omega)h| \geq \epsilon_0\}) \leq \frac{F_1(\epsilon_0)}{LA},$$

and plugging this into (3.17) gives us

$$|\nabla G(X)| \leq \frac{e^{\epsilon_0}}{\mathcal{H}^n(\partial\Omega)} + F_1(\epsilon_0) + C(n,M,L)\delta(X)^{\alpha}.$$

We can also bound $\delta(X)$ in terms of $G(X) = t = \epsilon_0$. An argument using Harnack chains and the Harnack Inequality shows that

$$G(X) \geq C\delta(X)^{\alpha'},$$

where C and $\alpha' > 0$ depend only on M and $\delta(0)$. Because $\delta(0)$ can be bounded below in terms of L, we get

$$\delta(X) \leq C(n, M, L)G(X)^{\frac{1}{\alpha'}}.$$

Thus

$$|\nabla G(X)| \leq \frac{e^{\epsilon_0}}{\mathcal{H}^n(\partial\Omega)} + F_1(\epsilon_0) + C(n, M, L)\epsilon_0^{\tilde{\alpha}},$$

with $\tilde{\alpha} > 0$.

Setting $F(\eta) = F_1(\eta) + C(n, M, L)\eta^{\tilde{\alpha}}$, we see that $F(\eta) \to 0$ as $\eta \to 0$ and

$$|\nabla G(X)| \leq \frac{e^{\epsilon_0}}{\mathcal{H}^n(\partial\Omega)} + F(\epsilon_0).$$

The function F has the properties stated in the theorem.

Let $\tilde{G}$ denote the Green's function for $B_{R_t}(0)$ with pole at the origin. By the comparison principle, we see that $\tilde{G} \leq G^t$ on $B(0, R_t)$, and we have $\tilde{G}(X) = G^t(X) = 0$. Therefore

$$|\nabla \tilde{G}(P)| \leq |\nabla G^t(P)| = |\nabla G(P)| \leq \frac{e^{\epsilon_0}}{\mathcal{H}^n(\partial\Omega)} + F(\epsilon_0).$$

We also know that

$$|\nabla \tilde{G}(X)| = \frac{1}{\sigma_n |X|^n} = \frac{1}{\sigma_n R_t^n} \geq \frac{1}{\sigma_n R^n},$$

where $R = \sup\{r > 0; B(0, r) \subset \Omega\}$, because $R_t \leq R$. Therefore

$$\frac{1}{\sigma_n R^n} \leq \frac{e^{\epsilon_0}}{\mathcal{H}^n(\partial\Omega)} + F(\epsilon_0).$$

Consequently,

$$R \geq \left[\sigma_n \left(\frac{e^{\epsilon_0}}{\mathcal{H}^n(\partial\Omega)} + F(\epsilon_0)\right)\right]^{-\frac{1}{n}},$$

as claimed.

□

3.4 The Outside Ball

For this part of the argument, we rely on the fact above, that Ω contains a relatively large ball. The idea then is to use Ahlfors regularity to show that the boundary $\partial\Omega$ cannot stray too far from the inner ball.

Theorem 4. *Let Ω, ω and h be as in Theorem 3, let $R_1 = \delta(0) = \sup\{r; B_r(0) \subset \Omega\}$, and let $R_2 = \inf\{r > 0; \Omega \subset B_r(0)\}$. There exists a constant $C > 0$ depending only on the parameters n, M, A, L and $\mathcal{H}^n(\partial\Omega)$ such that for sufficiently small ϵ_0 we have*

$$R_2 \leq R_1(1 + C\epsilon_0^{\frac{1}{2n^2}}).$$

Proof: Let $Q_0 \in \partial\Omega$ satisfy $|Q_0| = R_2$. Define a 'projection' on $\mathbb{R}^{n+1} \setminus \{0\}$ by

$$P(X) = \begin{cases} \frac{X}{|X|} R_1' & \text{if } |X| < R_1'' \\ \frac{X}{|X|} R_1'' & \text{if } |X| \geq R_1'' \end{cases},$$

where

$$R_1' = \left[\sigma_n \left(\frac{e^{\epsilon_0}}{\mathcal{H}^n(\partial\Omega)} + F(\epsilon_0)\right)\right]^{-\frac{1}{n}} \quad \text{and} \quad R_1'' = R_1' + \epsilon_0^{\frac{1}{2n}}.$$

Notice that this R_1' is the lower bound on the radius of the inner ball guaranteed by Theorem 3.

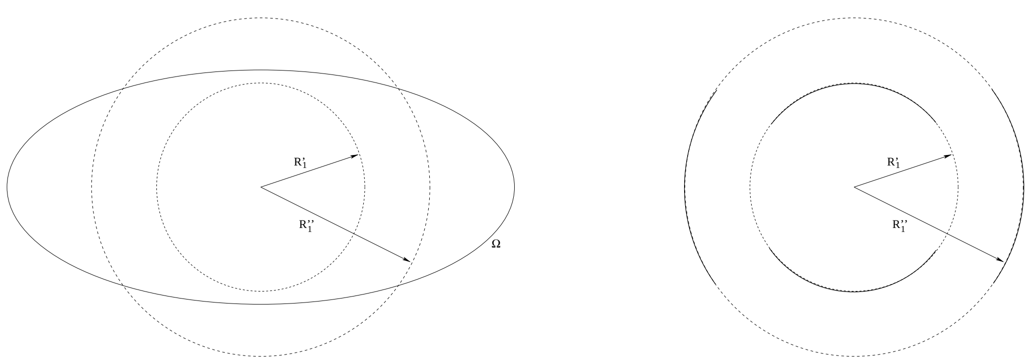

Figure 3.1: The darkened arcs on the figure at right represent the image under the projection P of the ellipse in the left figure.

The idea is to show that the part of $P(\partial\Omega)$ that gets projected onto $\partial B_{R_1''}(0)$ can't have large $\mathcal{H}^n$ measure, so that there isn't much of $\partial\Omega$ that's very far away from $\partial B_{R_1'}(0)$. Once we have control of the measure of this set, Ahlfors regularity will allow us to control the actual distance from $\partial B_{R_1'}(0)$.

Notice that the restriction of the projection to each set $B^c_{R''}$ and $B_{R''} \setminus B_{R'}$ satisfies $|P(X) - P(Y)| \leq |X - Y|$ (even though it does not have this property on their union); hence $\mathcal{H}^n(P(\partial\Omega) \cap B^c_{R''}) \leq \mathcal{H}^n(\partial\Omega \cap B^c_{R''})$ and $\mathcal{H}^n(P(\partial\Omega) \cap B_{R''}) \leq \mathcal{H}^n(\partial\Omega \cap B_{R''})$ (since $B_{R'} \subset \Omega$). Therefore

$$\mathcal{H}^n(P(\partial\Omega)) \leq \mathcal{H}^n(\partial\Omega).$$

Let

$$t = \frac{\mathcal{H}^n(P(\partial\Omega) \cap \partial B_{R_1''}(0))}{\mathcal{H}^n(\partial B_{R_1''}(0))}.$$

This is the fraction of $\partial B_{R_1''}(0)$ covered by the projection of $\partial\Omega$ under P. Then

$$1 - t \leq \frac{\mathcal{H}^n(P(\partial\Omega) \cap \partial B_{R_1'}(0))}{\mathcal{H}^n(\partial B_{R_1'}(0))}.$$

Thus we have

$$\begin{aligned}
\mathcal{H}^n(P(\partial\Omega)) &= \mathcal{H}^n(P(\partial\Omega) \cap \partial B_{R_1'}(0)) + \mathcal{H}^n(P(\partial\Omega) \cap \partial B_{R_1''}(0)) \\
&\geq (1-t)\sigma_n[R_1']^n + t\sigma_n(R_1'')^n \\
&= (1-t)\sigma_n[R_1']^n + t\sigma_n(R_1' + \epsilon_0^{\frac{1}{2n}})^n.
\end{aligned}$$

Therefore

$$
\begin{aligned}
\mathcal{H}^n(\partial\Omega) &\geq \mathcal{H}^n(P(\partial\Omega)) \\
&\geq (1-t)\sigma_n \left[\left(\sigma_n\left(\frac{e^{\epsilon_0}}{\mathcal{H}^n(\partial\Omega)} + F(\epsilon_0)\right)\right)^{-\frac{1}{n}}\right]^n \\
&\qquad\qquad + t\sigma_n \left[\left(\sigma_n\left(\frac{e^{\epsilon_0}}{\mathcal{H}^n(\partial\Omega)} + F(\epsilon_0)\right)\right)^{-\frac{1}{n}} + \epsilon_0^{\frac{1}{2n}}\right]^n \\
&= (1-t)\left(\frac{e^{\epsilon_0}}{\mathcal{H}^n(\partial\Omega)} + F(\epsilon_0)\right)^{-1} + t\left[\left(\frac{e^{\epsilon_0}}{\mathcal{H}^n(\partial\Omega)} + F(\epsilon_0)\right)^{-\frac{1}{n}} + \sigma_n^{-\frac{1}{n}}\epsilon_0^{\frac{1}{2n}}\right]^n \\
&\geq (1-t)\left(\frac{e^{\epsilon_0}}{\mathcal{H}^n(\partial\Omega)} + F(\epsilon_0)\right)^{-1} t\left(\frac{e^{\epsilon_0}}{\mathcal{H}^n(\partial\Omega)} + F(\epsilon_0)\right)^{-1} + \frac{t}{\sigma_n}\epsilon_0^{\frac{1}{2}} \\
&= \left(\frac{e^{\epsilon_0}}{\mathcal{H}^n(\partial\Omega)} + F(\epsilon_0)\right)^{-1} + \frac{t}{\sigma_n}\epsilon_0^{\frac{1}{2}}.
\end{aligned}
$$

In the second-to-last line we used the inequality $(a+b)^n \geq a^n + b^n$ for $a, b \geq 0$. We now have

$$
\mathcal{H}^n(\partial\Omega) \geq \left(\frac{e^{\epsilon_0}}{\mathcal{H}^n(\partial\Omega)} + F(\epsilon_0)\right)^{-1} + \frac{t}{\sigma_n}\epsilon_0^{\frac{1}{2}},
$$

and solving for t yields

$$
t \leq C\epsilon_0^{-\frac{1}{2}}\left[\mathcal{H}^n(\partial\Omega) - \left(\frac{e^{\epsilon_0}}{\mathcal{H}^n(\partial\Omega)} + F(\epsilon_0)\right)^{-1}\right]. \tag{3.18}
$$

This give us an upper bound on the measure of $P(\partial\Omega) \cap \partial B_{R_1''}(0)$ (more precisely, on the fraction, in terms of $\mathcal{H}^n$-measure, of the sphere $\partial B_{R_1''}(0)$ covered by $P(\partial\Omega)$).

The term in square braces is at most the order of ϵ_0 (which can be seen by looking at a Taylor series expansion of $a - \frac{1}{a+x}$ for x near 0), so we obtain

$$
t \leq C\epsilon_0^{\frac{1}{2}} \qquad \text{with } C = C(n, M, L, \mathcal{H}^n(\partial\Omega)).
$$

Now because the ball $B_{|Q_0|-R_1''}(Q_0)$ lies completely outside $B_{R_1''}(0)$, we get

$$\begin{aligned}
\frac{1}{A}(|Q_0| - R_1'')^n &\leq \mathcal{H}^n(\partial\Omega \cap B(Q_0, |Q_0| - R_1'')) \quad \text{(by Ahlfors Regularity)}\\
&\leq \mathcal{H}^n(\partial\Omega \setminus B(0, R_1''))\\
&= \mathcal{H}^n(\partial\Omega) - \mathcal{H}^n(\partial\Omega \cap B(0, R_1''))\\
&\leq \mathcal{H}^n(\partial\Omega) - \mathcal{H}^n(P(\partial\Omega \cap B(0, R_1'')))\\
&= \mathcal{H}^n(\partial\Omega) - \mathcal{H}^n(P(\partial\Omega) \cap \partial B(0, R_1'))\\
&\leq \mathcal{H}^n(\partial\Omega) - (1-t)\mathcal{H}^n(\partial B(0, R_1'))\\
&= \mathcal{H}^n(\partial\Omega) - (1-t)\sigma_n \left[\left(\sigma_n \left(\frac{e^{\epsilon_0}}{\mathcal{H}^n(\partial\Omega)} + F(\epsilon_0) \right) \right)^{-\frac{1}{n}} \right]^n\\
&= \mathcal{H}^n(\partial\Omega) - (1-t) \left(\frac{e^{\epsilon_0}}{\mathcal{H}^n(\partial\Omega)} + F(\epsilon_0) \right)^{-1}\\
&= t \left(\frac{e^{\epsilon_0}}{\mathcal{H}^n(\partial\Omega)} + F(\epsilon_0) \right)^{-1} + \left[\mathcal{H}^n(\partial\Omega) - \left(\frac{e^{\epsilon_0}}{\mathcal{H}^n(\partial\Omega)} + F(\epsilon_0) \right)^{-1} \right]\\
&\leq C\epsilon_0 \left(\frac{e^{\epsilon_0}}{\mathcal{H}^n(\partial\Omega)} + F(\epsilon_0) \right)^{-1} + \left[\mathcal{H}^n(\partial\Omega) - \left(\frac{e^{\epsilon_0}}{\mathcal{H}^n(\partial\Omega)} + F(\epsilon_0) \right)^{-1} \right]\\
&\leq C\epsilon_0 \mathcal{H}^n(\partial\Omega) + \left[\mathcal{H}^n(\partial\Omega) - \left(\frac{e^{\epsilon_0}}{\mathcal{H}^n(\partial\Omega)} + F(\epsilon_0) \right)^{-1} \right]\\
&\leq C\epsilon_0^{\frac{1}{2n}},
\end{aligned}$$

where in the last line we used the Taylor expansion again to see that the term in brackets is at most ϵ_0, which is less than $\epsilon_0^{\frac{1}{2n}}$ for small values of ϵ_0. So we now have

$$\frac{1}{A}(|Q_0| - R_1'')^n \leq C\epsilon_0^{\frac{1}{2n}}.$$

Hence

$$(|Q_0| - R_1'') \leq C\epsilon_0^{\frac{1}{2n^2}} \qquad \text{(for } C = C(n, M, L, \mathcal{H}^n(\partial\Omega), A)\text{)}.$$

Then since $|Q_0| = R_2$ and $R_1 \geq R_1' = R_1'' - \epsilon_0^{\frac{1}{2n}}$, we obtain

$$R_2 - R_1 \leq C\epsilon_0^{\frac{1}{2n^2}},$$

that is,

$$R_2 \leq R_1 + C\epsilon_0^{\frac{1}{2n^2}}.$$

Using the lower bound on R_1 obtained in Theorem 1, we can write this result in the form

$$R_2 \leq R_1(1 + C\epsilon_0^{\frac{1}{2n^2}}) \quad \text{(for } C = C(n, \mathcal{H}^n(\partial\Omega), A, L, M)\text{)}.$$

□

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VITA

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