Physics 222
Homework Assignment #1
Review of mechanics and the Fundamental Theorem of Calculus

1. This is a pure math problem about some special functions but you don't need to know any more about the functions than what is given here. You also shouldn't need to reach for Maple, MathCAD, or Derive to solve these. An ordinary scientific calculator and the fundamental theorem of calculus should be enough.

   The Gamma function $\Gamma(x)$ is a continuous and differentiable function for all positive values of x. It has the nifty property that for integer values of x, $\Gamma(x) = (x-1)!$ (that's a factorial symbol, not a statement that I'm excited). So $\Gamma(4) = 3! = 6, \Gamma(5) = 4! = 24,$ and so on.

   The Digamma function $\psi(x)$ is a continuous function defined as

   $$\psi(x) = \frac{d}{dx} \ln[\,\Gamma(x)]$$

   (which is probably just what you thought it would be).

   Using only a pocket calculator and the fundamental theorem of calculus, evaluate:

   $$\int_{7.0}^{11.0} \psi(x)\, dx$$

   SOLUTION:

   $$\int_{7.0}^{11.0} \psi(x)\, dx = \int_{7.0}^{11.0} \left(\frac{d}{dx} \ln[\,\Gamma(x)]\right) dx = \ln[\,\Gamma(x)]_{7.0}^{11.0} =$$
   $$ln[\Gamma(11.0)] - ln[\Gamma(7.0)] = ln[10!] - ln[6!] = 8.5$$

2. Another pure math problem. The logarithmic integral function $Li(x)$ is a continuous and differentiable function defined as

   $$Li(x) = \int_2^x \frac{1}{\ln(y)}\, dy$$

   Using only a pocket calculator and the fundamental theorem of calculus, find the slope of the function $Li(x)$ when *x = 2.3* (one short calculator calculation should do it).
   SOLUTTION:

   $$\frac{d}{dx}[Li(x)] = \frac{d}{dx}\left[\int_2^x \frac{1}{\ln(y)}\, dy\right] = \frac{1}{\ln(x)}$$
   $$\frac{1}{\ln(2.3)} = 1.2$$

3. A certain plook is a distance *x* away from a zoit. The potential energy of the plook-zoit system varies with the distance *x* and is given by

$$U(x) = a + b\,x + c\,x^2$$

Where *a = 1.73 joules*, *b = 3.52 joules/meter, and c = 11.2 joules/meter*^{2}.

a) Is the plook attracted to the zoit or repelled by it? Explain your reasoning.
b) What is the magnitude of the force between them when they are separated by a distance of 0.20 meters?
c) What piece of important-looking information in this problem is completely unnecessary to answer questions a and b?
d) Why is that piece of information useless? Explain in words.

SOLUTION:

a) The quadratic is purely positive, so as *x* increases the potential energy increases as well. Forces point toward *lower* potential energy. So the plook is attracted to the zoit. (And who wouldn't be?)
b) The (conservative) force is given by the negative of the derivative of the potential energy. Here we only need the magnitude so the minus sign doesn't matter.

$$F = -\frac{d}{dx}(a + b\,x + c\,x^2) = -(b + 2cx)$$

When *x* = 0.20 meters this gives us a force of 8.0 joules/meter = 8.0 newtons.
c) To solve this problem we didn't need to know "*a*" at all.
d) Potential energy by itself is not defined. Anyone can add or subtract an arbitrary constant to the potential energy of an object without changing anything physically measurable. The constant *a* in this problem is like the undefined "C" in an indefinite integral. It doesn't matter what that C is or what the potential energy is. The only thing that is measurable is the difference between the potential energies at two points.

4. A toy car begins from rest and is then subjected to a net force in the positive direction which increases at a steady rate from zero to 2.4 newtons in a period of 3.0 seconds. It then decreases back to zero at a steady rate in 1.0 seconds.
   a. What was the momentum of the toy car at the end of those 4.0 seconds?
   b. The final kinetic energy of the toy car was 14.4 joules. What was the mass of the car?

SOLUTION:

a) If you draw a graph of net force as a function of time, the plot forms a triangle above the horizontal axis. It slopes upward gently for three seconds and then rapidly downward for one second. The height of the triangle is 2.4 newtons. The length of the base is 4.0 seconds. The integral of net force as a function of time gives the change in momentum, and since momentum started at zero, this yields the final momentum.

$$p = \int \Sigma F\,dt = 0.5\,(base)(height) = 4.8\,Ns\ or\ 4.8\,kg\,m/s$$

b) $\frac{1}{2}mv^2 = \frac{1}{2m}(mv)^2 = \frac{p^2}{2m} = 14.4\,joules \qquad p = 4.8\,kg\frac{m}{s} \quad so\ m = 0.8\,kg$

5. A book is sliding across the floor under the influence of kinetic friction (which is constant). When the book starts sliding it has a speed of 7.0 m/s. It slows down as it travels, so when it reaches the far wall it has a speed of 1.0 m/s.
   a. State which happens first and explain your reasoning:
      i. the book reaches the point halfway across the room (in terms of distance) or
      ii. the book completes half of the journey in terms of time taken?
   b. What's the speed of the book when it has completed half of the journey in terms of time taken?
   c. What's the speed of the book when it has competed half of the journey in terms of distance traveled (when it is halfway across the floor)?

   (No you don't need to know the mass of the book or the magnitude of the friction force to solve problem #5. You only need to know that the net force on the book was constant.)

   SOLUTION:

   a) The book covers the first half of the distance at a higher average speed than the second half of the distance, so it reaches the midpoint in terms of distance ***before*** it reaches the midpoint in terms of time. Notice this means that the speed was greater at the midpoint in terms of distance than it was at the midpoint in terms of time.
   b) The net force on the book is constant, so a graph of momentum as a function of time will be a straight line. The mass is constant, so the same is true of a graph of velocity as a function of time. When half of the time has gone by, the velocity will be halfway between the initial velocity and the final velocity: $v = \frac{1}{2}\left(7.0\frac{m}{s} + 1.0\frac{m}{s}\right) = 4.0\frac{m}{s}$.
   c) What about when half of the distance has gone by? The work-energy theorem tells us that if the net force is constant then a graph of ***kinetic energy*** as a function of ***position*** will be a straight line. (You probably learned the work-energy theorem in integral form so you may need to think about this graph.) This means that the kinetic energy at the midpoint (in terms of distance) will be halfway between the kinetic energies at the start and at the end. Since the mass is constant, and kinetic energy is proportional to mass times the square of velocity, the same will be true of a graph of ***velocity squared*** as a function of position. The square of the velocity at the midpoint will be halfway between the squares of the initial and final velocities. What's halfway between the square of 1 and the square of 7? It's the square of 5: (49 + 1)/2 =25. So the velocity when half of the distance has been covered is

$$v = \sqrt{\frac{1}{2}\left(\left[7.0\frac{m}{s}\right]^2 + \left[1.0\frac{m}{s}\right]^2\right)} = \sqrt{\frac{1}{2}\left(49\ \frac{m^2}{s^2} + 1.0\frac{m^2}{s^2}\right)} = \sqrt{25\ \frac{m^2}{s^2}} = 5.0\ m/s$$

   Notice that this is faster than the speed when half of the time had expired, as predicted.